3-Manifolds

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Contents

1 Lecture 1: First Examples

Goal for this course: Understand the zoo of 3-manifolds to get lots of useful examples.

Themes:

- The most important invariant is π_1
- Looking for embedded surfaces $\Sigma \hookrightarrow M$ gives extra structure to M.
- $3 = 2 + 1$. Decomposing into a 2-dimensional structure and a 1-dimensional structure often helps.

We start by giving lots of examples and constructions of different 3-manifolds. The most basic ones are the sphere S^3, torus T^3, and a trivial circle bundle $\Sigma_g \times S^1$. We can adapt this last one to make a more interesting class of examples.

1.1 Mapping Tori

Suppose we have a fiber bundle $\Sigma_g \to M \to S^1$. Then we have an associated *monodromy representation* given by

$$\rho : \pi_1(S^1) \to \mathrm{MCG}(\Sigma_g)$$

which sends the generator of $\pi_1(S^1)$ to $[f]$ for some mapping class $[f]$. This monodromy defines M as a mapping torus given by

$$M_f = \Sigma_g \times [0,1] \Big/ \big((x,1) \sim (f(x),0)\big).$$

Proposition 1.1. *A closed, connected, oriented 3-manifold fibers over the circle if and only if it is a mapping torus M_f for a closed surface Σ_g and some $[f] \in \mathrm{MCG}(\Sigma_g)$.*

Proof. It is clear that a mapping torus fibers over the circle with fiber Σ_g. On the other hand, suppose that M fibers over the circle with fiber S.

$$\begin{array}{ccc} S & \xrightarrow{i} & M \\ & & \downarrow{\scriptstyle \pi} \\ & & S^1 \end{array}$$

Note that $S \cong \pi^{-1}(t)$ for a any point $t \in S^1$. It is then a closed subset of M, and S is compact. Since M has no boundary, S is compact without boundary.

Then cut along the surface $\pi^{-1}(0)$ to obtain a new 3-manifold with boundary M' which fibers over the interval $[0,1]$. Since the interval is contractible, every fiber bundle over this base is a product. Straighten out M' to make it a product. To get from M' to M, we glue via the monodromy of the fibration. This exactly constructs M as a mapping torus.[1] □

For the mapping torus, it is true that $f \sim h$ implies $M_f \cong M_h$ and it is even true if $[f]$ is conjugate to $[h]$ in $\mathrm{MCG}(\Sigma)$ then $M_f \cong M_h$. However, the converse to this statement is not true. There are $[f]$ and $[h]$ such that $M_f \cong M_h$ but $[f]$ and $[h]$ are not conjugate in $\mathrm{MCG}(\Sigma)$. In this case, the homeomorphism between M_f and M_h will not be fiber-preserving.

There is a version of this construction for surfaces with boundary. Let Σ_g^n be the genus g surface with n boundary components. Then M_f is a compact 3-manifold with ∂M_f a disjoint union of tori. To see this, we consider the fibration restricted to the boundary components. For each boundary component X of M_f, we have a fiber bundle $S^1 \to X \to S^1$. It is an orientable fiber bundle over the circle with circular fibers: so it must be a torus.

There is also a version for surfaces with punctures. Then we get M_f which is noncompact but is homeomorphic to the interior of of a mapping torus from the boundary case.

1.2 Fiber bundles over a surface

Examples include:

1. The Hopf fibration $S^1 \to S^3 \to S^2$.
2. The unit tangent bundle of a surface $S^1 \to UT(\Sigma_g) \to \Sigma_g$.

   Exercise: Show that this bundle is trivial if and only if $g = 1$.

3. A flat circle bundle. Let $\rho : \pi_1\Sigma_g \to \mathrm{Diff}^+(S^1)$ be a representation. Then $\pi_1\Sigma_g$ acts on the product space $\widetilde{\Sigma}_g \times S^1$. Let $\gamma \in \pi_1\Sigma_g$. The action is given by:

$$\gamma \cdot (z, \theta) = (\gamma \cdot z,\ \rho(\gamma)(\theta))$$

   Then we define the quotient space by this action $(\widetilde{\Sigma}_g \times S^1)/\pi_1\Sigma_g$.

[1] See this stack exchange post for more details.

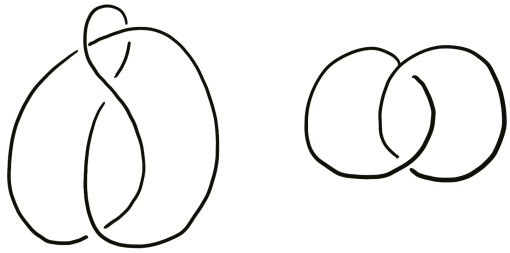

Figure 1: Two fibered links. On the left is the figure 8 knot, on the right is the Hopf link.

This is then a circle bundle over the surface. This construction is described more explicitly on page 4 of the taut foliations chapter in Calegari's 3-manifold book. We note that no points in the same circle fiber are glued together because ρ acts without fixed points on $\widetilde{\Sigma}_g$.

1.3 Knots in 3-manifolds

Let K be a knot in S^3. That is, a smooth embedding of the circle up to ambient isotopy.

Definition 1.2. Knots K and K' are *ambiently isotopic* if there is a homotopy through homeomorphisms on S^3

$$H : S^3 \times [0,1] \to S^3$$

such that $H(x,0) = x$ and $H(K,1) = K'$.

Remark 1.3. This definition is equivalent to the statement that there exists a $f \in \mathrm{Homeo}^+(S^3)$ such that $f(K) = K'$. This follows because $\mathrm{Homeo}^+(S^3)$ is path connected. That is, two knots are ambiently isotopic if and only if their complements are homeomorphic. (This is known as the Gordon-Luecke theorem.)

Then $M^3_K := S^3 \setminus K$ is a connected 3-manifold. The knot complement is actually the best knot-invariant! Knots are determined by their complement whereas links are not.

Question 1. *When does M^3_K fiber over S^1 with fiber a punctured surface? That is, which knots are fibered?*

Examples:

1. $S^3 \setminus K_8$ where K_8 is the figure 8 knot. The fiber is the once-punctured torus. This is not immeadiately clear.

2. $S^3 \setminus H$ where H is the Hopf link. To see this, we can take the Hopf fibration. Let $S^3 \subset \mathbb{C}^2$ be given by $S^3 = \{(z,w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\}$ and define an S^1 action given by
$$\theta \cdot (z,w) := (e^{i\theta}z, e^{i\theta}w).$$
Then $S^3/\text{this action} \cong \mathbb{C}P^1 = S^2$. If we take the pre-image of two distinct points in $\mathbb{C}P^1$ under this quotient, we get two linked circles as in the Hopf fibration.[2] Thus, $S^3 \setminus H$ is fibered by twice-punctured spheres.

Examples with algebraic surfaces.

Let $f \in \mathbb{C}[z,w]$ and look at the vanishing locus $Z(f)$. The vanishing locus is a complex 1-dimensional manifold (2-dimensional real manifold). Then f has a singularity when $\frac{\partial f}{\partial z} = 0 = \frac{\partial f}{\partial w}$. Around a singularity, we look at $Z(f) \cap S^3_\epsilon$ with S^3_ϵ a 3-sphere of small radius centered about the singularity. This is a 2-dimensional thing intersecting a 3-dimensional thing, so we get a closed 1-dimensional manifold... which is a link!

- If $f(z,w) = wz$, then we get the Hopf link!
- If $f(z,w) = z^3 - w^2$, then we get a trefoil knot. This tells us that the trefoil is a fibered knot.[3]

Exercise: See why these examples make sense with the given polynomial.

Theorem 1.4 (Stallings). *Recall that $H_1(S^3 \setminus K; \mathbb{Z}) \cong \mathbb{Z}$ and so there is a canonical map $\varphi : \pi_1(S^3 \setminus K) \to \mathbb{Z}$ where φ is the abelianization. Then $\ker(\varphi)$ is finitely generated if and only if K is a fibered knot.*

Exercise: Prove that $H_1(S^3 \setminus K; \mathbb{Z}) \cong \mathbb{Z}$ for any knot K.

[2] The proof of this fact uses the Euler class of the fibration to calculate the linking number.

[3] Also relevant: the story of how $\pi_1(S^3 \setminus T) \cong B_3 \cong \widetilde{\mathrm{SL}_2(\mathbb{Z})}$.

1.4 3-manifolds from geometry

Let X be a homogeneous Riemannian manifold so that there is a Lie group G which acts transitively by isometries. Then $X = G/K$ for K a compact subgroup of G. Some examples:

(a) $G = \mathrm{SO}(4)$, $K = \mathrm{SO}(3)$, $X = S^3$

(b) $G = \mathrm{Isom}^+(\mathbb{E}^3) = \mathbb{R}^3 \rtimes \mathrm{SO}(3)$, $K = \mathrm{SO}(3)$, $X = \mathbb{E}^3$

(c) $X = \mathbb{H}^3$, $G = \mathrm{PSL}(2, \mathbb{C})$, $K = \mathrm{SO}(3)$

(d) $X = \mathcal{N}i\ell \cong \mathbb{R}^3$, $N(\mathbb{R}) =$ Heisenburg group

Let $\Gamma < \mathrm{Isom}(X) = G$ be a discrete subgroup (this happens if and only if $\Gamma \circlearrowright X$ properly and discontinuously.) Assume that Γ acts freely on X (true iff Γ is torsion-free in the examples (b), (c), (d)). Then we have a closed Riemannian 3-manifold $\Gamma\backslash X = \Gamma\backslash G/K$.

Exercise: Fact: There are 10 different classes of flat closed orientable 3-manifolds. They can all be expressed in this $\Gamma\backslash G/K$ way. Find as many as you can!

Examples with the three different constant sectional curvature metrics:

- $\Gamma < \mathrm{SO}(4)$ finite group $\Rightarrow$ lens spaces
- $\lambda < \mathbb{R}^3$ full rank lattice $\Rightarrow \lambda\backslash\mathbb{E}^3 \cong T^3$ is a 3-torus
- Let
$$\Gamma = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix} \right\rangle < \mathrm{PSL}_2\,\mathbb{Z}[\omega] < \mathrm{PSL}(2, \mathbb{C}).$$
Note that Γ is order 12 in $\mathrm{PSL}_2\,\mathbb{Z}[\omega]$. Then $\mathbb{H}^3/\Gamma$ is a non-compact 3-manifold with finite volume and curvature $\equiv -1$. It's single end has a torus neighborhood.

Theorem 1.5 (Riley 1974). *For the Γ above, $\mathbb{H}^3/\Gamma \cong S^3 \setminus K_8$ and has the structure of a fiber bundle $(T^2 - \{pnt\}) \to M \to S^1$.*

Thurston proved that every knot which is not satellite or a torus knot has a complete hyperbolic metric on the complement. We will return to this result later in the quarter.

2 Lecture 2: Geometrization Conjecture Statement...

... and surfaces in $S \subset M^3$.

First, another example and an exercise from last time on the Heisenburg group. Let $H(G)$ be the upper-triangular matrices with 1 on the diagonal and entries in G.

$$H(G) := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle| x, y, z \in G \right\}$$

Then we define $M := H(\mathbb{R})/H(\mathbb{Z})$ and note that $\pi_1(M) \cong H(\mathbb{Z})$ by construction.[4] This 3-manifold M is the only one which fibers in two different ways. Ask Benson about this? Maybe he means both over a circle and over a surface?

Exercise:

(a) Prove that M fibers as $T^2 \to M \to S^1$ with monodromy $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$$1 \to \mathbb{Z} \to H(\mathbb{Z}) \to \mathbb{Z}^2 \to 1$$

with $\mathbb{Z}$ generated by matrix with $x = y = 0$ and $\mathbb{Z}^2$ generated by matrices with only x, y nonzero respectively.

(b) Prove that M fibers as $S^1 \to M \to T^2$. It is a non-trivial S^1-bundle with Euler class = 1.

2.1 Spheres in M^3

Definition 2.1. If M and N are two 3-manifolds, then the *connected sum* $M_1 \# M_2$ is given by

$$M_1 \# M_2 := (M_1 \setminus B_1) \sqcup (M_2 \setminus B_2)$$

where the boundaries of $M_i \setminus B_i$ are identified with an orientation reversing homeomorphism.

Remark 2.2. –

[4] I suppose you also need to know that $H(\mathbb{R})$ is contractible.

1. The above definition is invariant of the choice of such a homeomorphism because $\mathrm{MCG}(S^2)$ is trivial.

2. Notice that there is then a *separating embedded sphere* $S^2 \hookrightarrow M_1 \# M_2$. If neither M_1 or M_2 are S^3, then this 2-sphere is $\neq 0$ in $\pi_2(M_1 \# M_2)$.

3. Note that $\pi_1(M_1 \# M_2) \cong \pi_1(M_1) * \pi_1(M_2)$ because $\pi_1(S^2) = 0$.

Remark 2.3. **Two finicky points about embeddings/immersions of spheres:**

- Q1: Given a non-trivial immersed S^2 in M, can we represent its homology class with an *embedded* S^2? This question is answered by something called the sphere theorem proved by Papakyriakopoulos in 1957.

- Q2: Given an embedded S^2 which is homotopically trivial, does it bound an embedded ball? This is answered (in the affirmative?) by the Poincarè theorem.

Definition 2.4. A 3-manifold M^3 is *prime* if whenever $M^3 = A \# B$, either $A \cong S^3$ or $B \cong S^3$.

Theorem 2.5 (Prime decomposition Theorem, Kneser 1920s-30s; uniqueness, Milnor)**.** *Every compact 3-manifold M has a prime decomposition*

$$M = M_1 \# \dots \# M_r$$

with each M_i prime. This decomposition is unique up to homeomorphism and permutation of the factors.

proof idea for existence. If M is not prime, then $M = M_1 \# M_2$ and $\pi_1 M \cong \pi_1 M_1 * \pi_1 M_2$. By the Poincaré Theorem, $\pi_1 M_i \neq 0$ since M_i are not homology spheres. But Gusko's theorem says that $\mathrm{rank}(A * B) = \mathrm{rank}(A) + \mathrm{rank}(B)$ where the rank is the minimum number of generators. So we have $\mathrm{rank}(M_i) < \mathrm{rank}(M)$. But this process will eventually stop because $\mathrm{rank}(M) < \infty$ since M is compact. Once it terminates, we have a prime decomposition. □

Definition 2.6. M is *irreducible* if every embedded $S^2 \hookrightarrow M$ bounds an embedded ball.

Remark 2.7. Notice that (Irreducible) $\Rightarrow$ (Prime) and that (Prime) $\Rightarrow$ (Irreducible or $M = S^2 \times S^1$). We can then often reduce the study of 3-manifolds to looking at primes manifolds and the special case of $S^2 \times S^1$.

2.2 Tori in M^3

Theorem 2.8 (Geometrization, Perelman 2001)**.** *For every compact irreducible M^3, there is a "canonical" possibly empty collection of disjoint embedded tori $T_1, \dots, T_r$ such that*

(i) Each T_i is incompressible. Meaning that $\iota_* : \pi_1(T_i) \to \pi_1(M^3)$ is injective,[5]

(ii) Each path component of $M^3 \setminus \bigcup T_i$ admits a locally homogeneous, complete Riemannian metric of finite volume.

Remark 2.9. *Locally homogeneous* means that every point has a neighborhood which are all isometric. By Singer, if M is locally homogeneous, then $\widetilde{M}$ is homogeneous (i.e. there is an isometry of the whole space taking any point to any other).

Thurston showed that each component of $M^3 \setminus \bigcup T_i$ is given by X/Γ where X is isometric to one of 8 geometries and Γ is a discrete subgroup of isometries. The 8 geometries are:

$$\mathbb{E}^3, \quad \mathbb{H}^3, \quad S^3, \quad S^2 \times \mathbb{E}^1, \quad \mathbb{H}^2 \times \mathbb{E}^1, \quad Nil, \quad Sol, \quad \widetilde{\mathrm{PSL}(2, \mathbb{R})}$$

An example of $\mathcal{N}i\ell$ is the Heisenburg manifold described at the beginning of this lecture. An example of $\mathcal{S}o\ell$ is a mapping torus of T^2 with Anosov monodromy (i.e. a fiber bundle $T^2 \to M \to S^1$ where the monodromy is given by $A \in \mathrm{SL}_2 \mathbb{Z}$ with $|\operatorname{tr} A| > 2$). For $\widetilde{\mathrm{PSL}(2, \mathbb{R})}$, the unit tangent bundle of a genus $g \geq 2$ surface $UT(\Sigma_g)$ has this geometry.

Exercise: Think about these examples and maybe look up more things about them.

Remark 2.10. There is a (unmentioned) complete invariant which determines the geometry and decomposition.

Corollary 2.11. *If M is a closed, irreducible, atoroidal 3-manifold, and if $|\pi_1 M| = \infty$ and M is not a Seifert fibered space, then M is hyperbolic and $M \cong \mathbb{H}^3/\Gamma$ for $\Gamma < \mathrm{Isom}^+(\mathbb{H}^3)$.*

Remark 2.12 (Hyperbolic Geometry Remark)*.* –

[5] This is actually a slightly different but equivalent notion to incompressible. The true definition is that there is no nontrivial curve γ in T_i such that γ bounds a disk in M. In our case, these are the same.

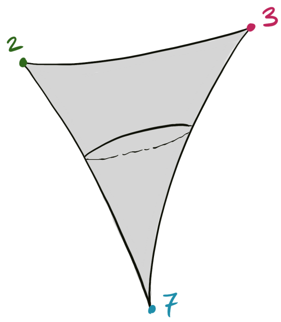

Figure 2: The sphere with three cone points of orders $2, 3, 7$. This is a hyperbolic orbifold.

1. If M is a closed hyperbolic n-manifold with $n \geq 2$, then $\mathbb{Z}^d < \pi_1(M)$ implies $d = 1$ and $\pi_1(M)$ is infinite and torsion free.

2. Suppose we have hyperbolic manifold $M = \mathbb{H}^3/\Gamma$ with Γ torsion-free so that M has a finite volume complete hyperbolic metric. We can decompose M along tori to "cut off" the cusps. This is called the thick-thin decomposition. See Section 9.2.

Exercise: Prove fact 1 in the remark. Look at two commuting elements in the π_1 and how they would act on $\mathbb{H}^n$. Should get that these should both be hyperbolic translations with the same axis. (Ping Pong!)

2.3 Small Seifert fibered space

A *Seifert fibered space* is a circle bundle over an orbifold. If the orbifold has no embedded essential loops, then it is a *small* Seifert fibered space.

$$S^1 \to M \to \mathcal{O}$$

As as example, we look at the orbifold $\mathcal{O}$ which is the sphere with three cone points of orders 7, 3, and 2 (see Figure 2). And we let the manifold M be the unit tangent bundle $UT(\mathcal{O})$. A presentation for $\pi_1(\mathcal{O})$ is given by

$$\pi_1(\mathcal{O}) = \langle a, b \mid a^2 = b^3 = (ab)^7 = 1 \rangle$$

And the 3-manifold group $\pi_1(M)$ is given by

$$\pi_1(M) = \langle a, b, t \mid [t, a], [t, b], a^2 = b^3 = (ab)^7 = t \rangle$$

There are no embedded *simple* closed curves in $\mathcal{O}$, and so we get no *embedded* essential tori in $M = UT(\mathcal{O})$. Examples like M and other small Seifert fibred spaces are why we need the extra stipulation in Corollary 2.11.

3 Lecture 3: Geometrization and Equivalence to Andreèv's theorem

Theorem 3.1 ((Hyperbolization theorem)Geometrization, Perelman 2001)**.** *Let M be a closed 3-manifold. Then M has a hyperbolic structure unless:*

1. *M has an essential 2-sphere (which will happen if and only if $\pi_2(M) \neq 1$ thanks to the Poincaré theorem). In other words, M has an embedded S^2 which does not bound a 3-ball.*
2. *$\mathbb{Z}^2 < \pi_1(M)$. This happens if and only if there is an embedded π_1-injective T^2 or M is a small Seifert fibered surface space.*
3. *$|\pi_1(M)| < \infty$.*

Exercise: Show that hyperbolic manifolds fail 1, 2, 3.

Proof idea: construct a hyperbolic structure on M and the construction will fail exactly in cases 1, 2, 3.

3.1 Right-angled hyperbolic polyhedra

Definition 3.2. A combinatorial polyhedra A is given by an embedded trivalent connected graph Γ in S^2 such that

(i) complimentary regions of the graph are topological disks.

(ii) the dual graph has no self-loops.

(iii) the dual graph has no double edges.

The graph Γ is the 1-skeleton of ∂A.

Question 2. *What are the necessary and sufficient conditions for when we can realize this as the edge graph of a convex right-angled hyperbolic polyhedron?*

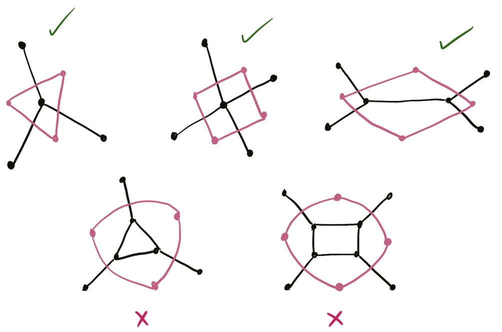

Figure 3: A list of allowed and disallowed cycles for the graph to satisfy the conditions of Andreèv's theorem.

Remark 3.3 (Ideal vertices)*.* There is a generalization in which we can allow some 4-valent vertices of the graph. In this case, we will seek to realize A as a semi-ideal hyperbolic polyhedron P. That is, a polyhedron with some vertices on the sphere at infinity (*ideal vertices*) and some vertices in the interior of $\mathbb{H}^3$ (*finite vertices*)

Finite vertices have a link[6] which is part of a 2-sphere. If you add up the dihedral angles around that vertex, there must be a total angle of $< 2\pi$. Each dihedral angle will be $\frac{\pi}{2}$ and so we cannot have more than 3-valent finite vertices.

However, for an ideal vertex, the link will be a horosphere centered at that vertex, and so the angle sum will be 2π. This allows for 4-valent ideal vertices.

3.2 Andreèv's theorem

Theorem 3.4 (Andreèv's theorem (maybe with Koehe and Thurston))**.** *Let A be a combinatorial polytope with vertices of valence 3 and 4. Then A admits a unique*[7] *right-angled semi-ideal hyperbolization unless:*

1. *there is a 3-cycle in the dual graph which is not the link of a 3-valent vertex.*
2. *there is a 4-cycle in the dual graph which is not the link of a 4-valent vertex or the link of an edge or A is a square-based pyramid.*
3. *A is the tetrahedron.*

[6]in the combinatorial sense
[7]up to isometry

Remark 3.5. We will see that 1, 2, and 3 in the two theorems (3.1 and 3.4) correspond to each other. Andreèv's theorem proves the geometrization conjecture in the special case where we have a 3-manifold constructed by gluing hyperbolic polyhedra.

Here is the correspondence between the two theorems. Given a combinatorial polyhedron A, declare that A is a topological orbifold. Now suppose that ∂A has n facets. Note that

$$2^n = |\{\text{functions } f : \{\text{facets of } A\} \to \{0,1\}\}| .$$

Then take 2^n copies of A, each with a different labeling of the facets by 1 or 0. Then for any copy A_0 and facet σ of A, there is some other copy A_1 which has identical labeling as A_0 except it differs on σ. Glue these two polyhedra together along the face σ. About an edge, there will be 4 polyhedra glued together. About a vertex, there will be 8.

Exercise: Show that for this construction,

1. a 3-cycle which is not the link of a vertex will produce an essential sphere after gluing;
2. a 4-cycle which is not the link of something will produce an embedded essential torus;
3. the one exceptional case is that of 16 right-angled tetrahedra which will produce S^3 cut by the 3 coordinate planes.

Remark 3.6. The geometrization conjecture implies Andreèv's theorem. And Andreèv's theorem implies the geometrization conjecture for the special case of hyperbolic 3-manifolds which are M^3/Γ for Γ a discrete subgroup of isometries.

4 Lecture 4: Proof of Andreèv's theorem I

We will prove Andreév's Theorem. The outline of the proof goes as follows:

1. Build a polyhedron $\hat{A}_0$ based on the combinatorial polytope given.
2. For each $\nu \in [0, \frac{\pi}{2}]$, we define a $\hat{A}_\nu$ which has dihedral angles ν unless something bad happens which corresponds to 1, 2, or 3. We want to start with $\hat{A}_0$ and deform it into $\hat{A}_{\frac{\pi}{2}}$ by taking ν from 0 to $\frac{\pi}{2}$.
3. Then $\hat{A}_{\frac{\pi}{2}}$ will be the desired object which we look for.

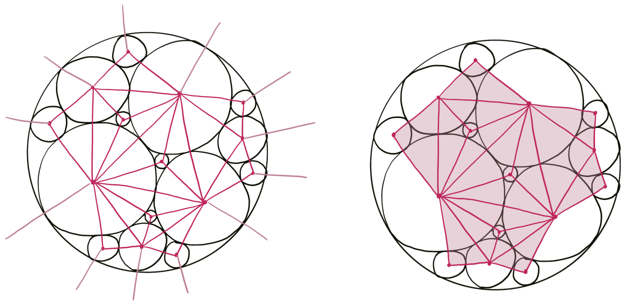

Figure 4: A circle packing in $\mathbb{H}^2$ and a corresponding triangulation. On the right we have highlighted the polygon P.

4.1 Circle Packings

Let τ be a triangulation of S^2. Then a *circle packing* associated to τ is a collection of round circles in S^2 which are mutually tangent in a pattern corresponding with τ. The interiors of the circles will be centered on vertices of τ and the tangencies of circles correspond to edges between vertices. See Figure 4 for an example.

Remark 4.1. We want τ to be a *triangulation* of the sphere instead of a general cell decomposition so that the circle packing is rigid up to conformal automorphism (Möbius transformations).

Theorem 4.2. *Every triangulation τ of S^2 corresponds uniquely to a circle packing up to conformal automorphism.*

Notice that we can rotate the sphere to center any given circle at the north pole by composing with a conformal automorphism. Then we can view the circle packing as a circle packing in $\mathbb{H}^2$ containing S^1_∞ as one circle, along with horocycles and other circles in the interior. So we can reduce to looking at circle packings in the hyperbolic disk.

proof of 4.2. Given τ and a choice of vertex v_0 to center at the north pole, we get an associated polygon P with a triangulation by taking $\tau \setminus v_0$ (i.e. remove vertex v_0 and all edges connecting it). By induction, it suffices to assume that there are no two boundary vertices of P connected by an edge through the interior of the polygon. We will show that there is a circle packing of $\mathbb{H}^2$ consisting of circles and horocycles where the tangencies correspond to the edges in the triangulation of P.

Let $V(P)$ be the set of interior vertices in P and $V = |V(P)|$. Let $\mathcal{L}$ be the functions ℓ from $V(P)$ to $[0, \infty)$ which satisfy a certain property (Q). For $i \in V(P)$, we think of $\ell(i)$ as being the hyperbolic radii of the circle centered at i in the attempted packing. As we "wiggle" the function ℓ, we change the radii of the circles. We want to adjust these to be exactly the right length so we obtain the appropriate tangencies.

For a triangle with vertices i, j, and k, the side lengths are determined by the function ℓ as $\ell(i) + \ell(j)$, $\ell(j) + \ell(k)$, and $\ell(k) + \ell(i)$. The side lengths determine a hyperbolic triangle up to isometry and so the angles at the vertices i, j, k of this triangle are well-defined. A function ℓ has property (Q) if: *the angle sum around each vertex is* $\geq 2\pi$.

And so we define a map on $\mathcal{L}$.

$$\text{excess} : \mathcal{L} \to [0, \infty), \qquad \ell \mapsto \sum_{v \in V(P)} (\text{total angle at } v - 2\pi)$$

If excess$(\ell) = 0$, then ℓ is a legitimate circle packing. That is, if $\ell \mapsto 0$, then there is no holonomy around any vertex and the order of "laying down the circles" does not matter. We claim that there is some $\ell \in \mathcal{L}$ such that excess$(\ell) = 0$. This follows from four statements.

1. If $\ell \mapsto \alpha > 0$, then we can modify ℓ to make excess(ℓ) smaller.
2. $\mathcal{L}$ is compact.
3. excess is a continuous function.[8]
4. Given a polygon P (with enough vertices), $\mathcal{L}$ is non-empty.

Remark 4.3. This implies that we have a "local circle packing", i.e. we can lay out all the circles tangent to any fixed circle and it will work out. The fact that if things lay out locally, then it works out globally follows from Poincaré's polyhedron theorem.

We address each of these claims independently.

1. For 1, if we increase $\ell(i)$ a little bit, then we increase the area of any triangle Δ containing vertex i. By Gauss Bonnet, Area$(\Delta) = \pi - \alpha - \beta - \gamma$, and so some angle of the triangle Δ decreases. Thus, the total angle excess decreases. See Figure 5.
2. For 2, we see that $\mathcal{L}$ is a subspace of $[0, \infty)^V$, the space of all functions from $V(P)$ to $[0, \infty)$. So we argue that $\mathcal{L}$ is a closed and bounded subspace. First, it is closed because the $\geq 2\pi$ is a closed condition. Second, it is bounded because if we take $\ell(i)$ to ∞ for a fixed i, then ℓ leaves $\mathcal{L}$. There are two ways to

[8]the topology is the standard product topology on $\oplus \operatorname{Hom}(\{\text{pt}\}, \mathbb{R}^+)$

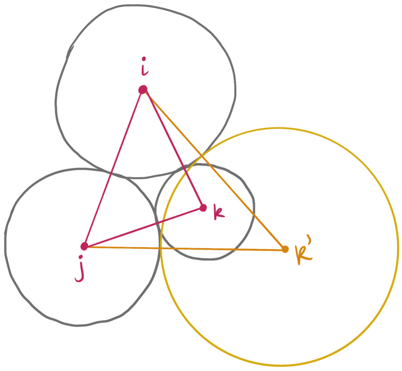

Figure 5: Increasing the radius of some circle increases the area of an associated triangle.

see this: if we make a circle centered at vertex i too large, then the circles around it loose their tangency relations. Another way to think about it is that if a hyperbolic triangle has two sides with length arbitrarily large, their shared angle gets arbitrarily close to 0. This violates property (Q) for that vertex.

4. For 4, we do induction on the number of vertices of P. Take the graph τ and delete any vertex, then triangulate the resulting face without adding in an extra vertex. This gives a triangulation τ' with one fewer vertex. Then center the face from which you deleted the vertex at ∞. Use the induction assumption to create a circle packing for τ'. Then grow the circles centered at the vertices around the face centered at ∞ into horocycles. An example of this process is shown in Figure 6.

Exercise: Prove claim 3.

□

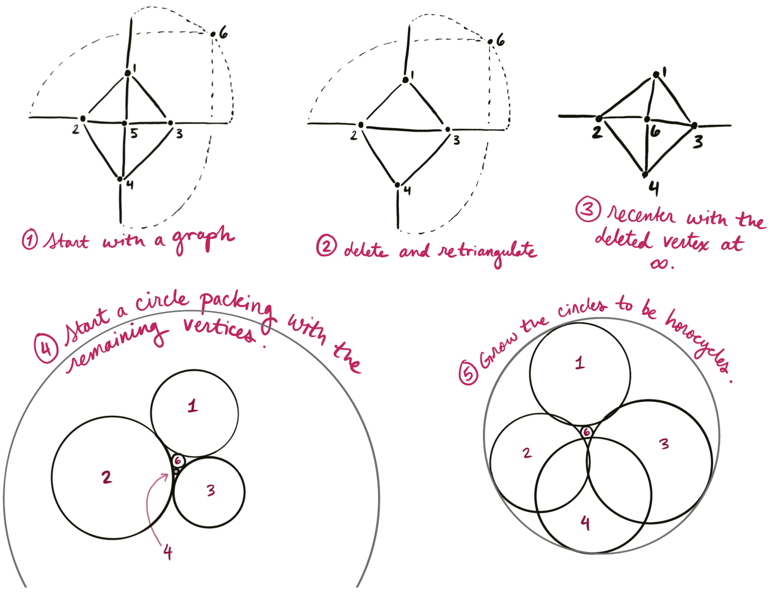

Figure 6: An example of the induction step to show that $\mathcal{L}$ is nonempty.

Where we are going after this:

Given a circle packing, we also construct the dual packing. We look at the union of these two circle packings in S^2 and view the sphere as S^2_∞ of $\mathbb{H}^3$. Each circle bounds a hemisphere in the interior of $\mathbb{H}^3$. Two hemispheres will either be tangent on S^2_∞ (if they are in the same circle packing), or meet in the interior of $\mathbb{H}^3$ at right angles (if they are in dual circle packings). Together, the hemispheres bound a right-angled polyhedron in $\mathbb{H}^3$.

5 Lecture 5: Proof of Andreèv's theorem II

Recall the statements of the Hyperbolization theorem 3.1 and Andreèv's theorem 3.4. We will continue the proof of Andreèv's theorem and let this suffice for proof of Hyperbolization.

Proof outline. –

1. Let $\Gamma \subset \partial A$ be the 1-skeleton and let Γ' be the dual graph. Then Γ' will have faces which are triangles and quads corresponding the the valence 3 and 4 vertices, respectively.

   Then we find a circle packing on S^2 corresponding with Γ' such that for all quads, all points of tangency lie on a circle. Note that this condition makes the

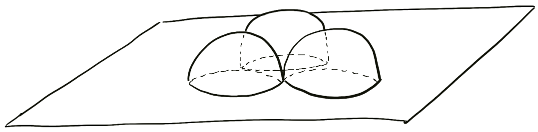

Figure 7: Hemispheres sitting above the plane with boundary as the circles in the circle packing. We think of these as geodesic hyperplanes in $\mathbb{H}^3$.

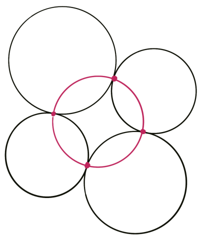

Figure 8: Pictures of four circles around a quad. The four points of tangency lie on a circle.

packing rigid up to conformal isometry for the places where 4 circles meet (see Figure 8).

2. We find a 1-parameter family of ν-packings for Γ'. The first one we find is for $\nu = 0$. The number ν measures the angle of intersection of the circles.

3. We increase ν from 0 to $\frac{\pi}{2}$. Then we think of the circles on S^2 as the boundaries of totally geodesic planes in $\mathbb{H}^3$. (They look like bubbles sitting above a plane. See figure 7.)

   The problems to solve are: (a) How do we find a ν-packing? (b) How do we deform the packing to increase ν a little?

   For the fixed graph Γ, we define $U \subset [0, \pi]$ to be the set of ν for which there is a ν-packing. Then we prove that either U is both open and closed, or one of the "bad things" in the statement of the theorem happens (i.e. numbers 1-3 in Theorem 3.4).

Consider what happens as we increase ν around a vertex of Γ/face of Γ'. We track an orthocricle which intersects each of the 3 or 4 circles orthogonally. As ν increases, this circle shrinks. For 3 circles, the critical moment is when $\nu = \frac{\pi}{3}$ where the circle shrinks to a point. For 4 circles, this happens when $\nu = \frac{\pi}{2}$.

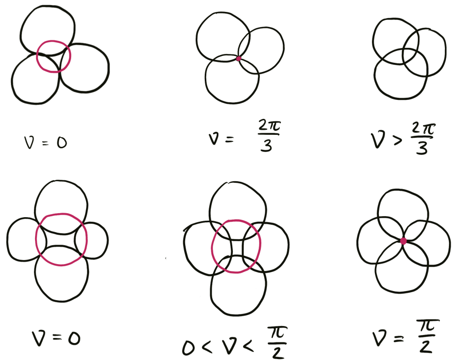

Figure 9: Vertices of valence 3 and valence 4 as ν increases from 0.

Then we construct A_ν as the complement of the planes bounded by circles and orthocircles in $\mathbb{H}^3$. Note that the intersection of a circle and an orthocircle is $\frac{\pi}{2}$ by construction, and the intersection of two non-orthocircles is ν. So as we increase ν to $\frac{\pi}{2}$, we make all of our angles $\frac{\pi}{2}$.

The reason why we add in the orthofaces is because we want A_ν to have finite volume at every stage.

Claim 5.1. *The volume of $\hat{A}_\nu$ is monotonically decreasing as a function of ν as $\nu \to \frac{\pi}{2}$.*

Then, in order to apply the claim, we need a nice 1-parameter family of polyhedra. There are two ways which the family could degenerate. (i) The planes get "very far apart" or (ii) the planes "collapse and get small".

In case (i), we can rule out the situation where we have 2 faces getting far apart because of a volume bound (?). In this case we end up with the tetrahedran and square-based pyramid.

The claim above follows from the Schläffli formula.

Proposition 5.2 (Schläffli formula)**.** *Let $P(t)$ be a 1-parameter family of hyperbolic polyhedra. Then*

$$\frac{d\,\mathrm{Vol}(P(t))}{dt} = -\frac{1}{2}\sum_{e\ edge} length(e)\frac{dangle(e)}{dt}.$$

Note that the edges which are the intersection of one circle plane and one orthocircle plane have a constant angle at $\frac{\pi}{2}$ by construction. The other edges have an angle of ν and so the change is $\frac{d\nu}{d\nu} = 1$. If we apply the Schläffli formula then to our situation, we have that $\frac{d\,\mathrm{Vol}(P(t))}{dt}$ is negative. And thus, the volume is monotonically decreasing.

Proof sketch of Schläffli. The proof is a probabilistic one. We start with the *Crafton formula* which says that $\mathrm{Vol}(P) = C_n \,\mathrm{Area}(P \cap \pi)$ where π s a random totally geodesic hyperplane. And note that the intersection $P \cap \pi$ is a hyperbolic polygon. We then apply the Gauss Bonnet theorem, and we see that increasing angles decreases the area. □

□

Remark 5.3. –

- Recovers the idea of how to find hyperbolic structures on 3-manifolds.
- Thurston's proof for Haken manifolds uses Andreev's theorem as a base case.
- We can prove hyperbolization for an orbifold in an analogous way

I lost the end of this lecture and proof. It could be worth it to go read Danny's write up on this proof and try to figure out the details.

6 Lecture 6: Mostow Rigidity using Gromov Norm I

> Mathematicians: "We think that isometry type is an invariant of homotopy for hyperbolic n-manifolds with $n \geq 3$."
>
> Gromov: "I will look for a homotopy invariant which happens to be equal to volume for constant -1 curvature manifolds. Then I am halfway to proving that the whole isometry type is invariant of homotopy."

6.1 Defining Gromov norm

Let X be a topological space and we look at the singular i-chains on X.

$$C_i(X) \otimes \mathbb{R} := \{\text{singular chains on } X\} = \left\{ \sum_{i=1}^{r} a_i \sigma_i \,\middle|\, a_i \in \mathbb{R},\ \sigma_i : \Delta^i \to X \right\}$$

This is a real-vector space with a *canonical (uncountable) basis*. We put the L^1 norm on this space.

$$\left\| \sum a_i \sigma_i \right\| := \sum |a_i|$$

Exercise: Check that this is a norm.

Definition 6.1. The *Gromov norm* on the i^{th} homology is $\|\cdot\| : H_i(X) \to [0, \infty)$ defined by

$$\|\xi\| := \inf\{\|c\| : c \text{ is an } i\text{-cycle and } [c] = \xi\}$$

In words, the Gromov norm can be described as *the least number of simplices which are needed to represent a given homology class.*

Proposition 6.2. *If $f : X \to Y$ is a continuous map, then for all $\xi \in H_i(X)$, we have $\|f_*\xi\| \leq \|\xi\|$.*

Proof. If $\xi = [\sum a_i \sigma_i]$, then $\|f_*\xi\| \leq \|\sum a_i(f \circ \sigma_i)\| \leq \sum |a_i|$. Note that we do not have equality in the last statement exactly because some of the maps $f \circ \sigma_i$ may not be distinct even though the σ_i are distinct. □

By Poincarè Duality, if M is a closed, connected, oriented n-manifold, then $H_n(M; \mathbb{Z}) \cong \mathbb{Z}$ (canonically) and is generated by $[M]$.

Definition 6.3. The *Gromov norm* of a manifold M is the Gromov norm of its fundamental class: $\|M\| := \|[M]\|$.

Corollary 6.4. *If $f : M \to N$ is a homeomorphism, then $\|M\| = \|N\|$.*

Example: We can realize the circle with a $\frac{1}{n}$ coefficient because we can map a 1-simplex in and wrap it around the circle n times. Since we can do this for any n, we get that $\|S^1\| = 0$.

Proposition 6.5. *For a continuous map $f : M \to N$, we have $\|M\| \geq \deg(f)\|N\|$.*

This proposition follows directly from the definition of degree. We have that the degree is the exact number so that $f_*[M] = (\deg f)[N]$. Any so we just push forward any cellulation of $[M]$ to get the right inequality.

Corollary 6.6. *If there is some $f : M \to M$ such that* $\deg f > 1$, *then* $\|M\| = 0$.

By consequence, we see that $\|T^n\| = 0$ and $\|S^n\| = 0$ for every $n \geq 1$.

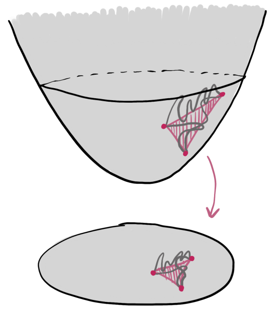

Figure 10: The simplex straightening map for $n = 2$.

6.2 Simplex Straightening

Let M be a closed hyperbolic n-manifold given by $\mathbb{H}^n/\Gamma$ where Γ is the image of $\rho : \pi_1(M) \to \text{Isom}^+(\mathbb{H}^n)$. Then we define a straightening map $\text{str} : C_i(M) \to C_i(M)$ in the following way.

1. Given $\sigma : \Delta \to M$, pick some lift of the map $\tilde{\sigma}$ to the universal cover $\mathbb{H}^n$.

2. Take the Euclidean simplex with those vertices in the hyperboloid model. See Figure 10 for an example when $n = 2$.

3. Project this simplex down to the hyperboloid. Define a continuous map from $\tilde{\sigma}$ to this projected simplex.

4. Compose with the quotient map to get a chain in M.

There are two key properties of the straightening map which make it particularly nice.

(i) str is $\text{Isom}^+(\mathbb{H}^n)$-equivariant. This means that for all $\varphi \in \text{Isom}^+(\mathbb{H}^n)$, we have $\text{str}(\varphi \circ \sigma) = \varphi \circ \text{str}(\sigma)$.

(ii) str is chain-homotopic to the identity through straight-line homotopy.

So str induces a map on homology $\text{str} : H_i(M^n) \to H_i(M^n)$. And we also have $\|\text{str}(c)\| \leq \|c\|$. We don't only have equality here because two "floppy maps" can give the same straight map so the coefficients can cancel out.

Exercise: Think about why str induces a map on homology.

Theorem 6.7 (Gromov's volume theorem)**.** *Let $n \geq 2$. Then there is some $v_n > 0$ depending on the dimension such that for all closed hyperbolic n-manifolds M,*

$$\|M\| = \frac{\mathrm{Vol}(M)}{v_n} > 0.$$

Corollary 6.8. *If M^n is a closed hyperbolic manifold, then M^n does not admit self-maps of degree greater than 1.*

Proof. If there was such a map, then by Corollary 6.6, $\|M\| = 0$. But a hyperbolic n-manifold has positive volume, so this contradicts Theorem 6.7. □

Proof of $\geq$ for Theorem 6.7. Let v_n be defined as

$$v_n = \sup\{\mathrm{Vol}(\sigma) \ : \ \sigma : \Delta^n \to \mathbb{H}^n \text{ is straight}\}$$

where volume is given by $\mathrm{Vol}(\sigma) = \int_{\Delta^n} \sigma^* d\,\mathrm{Vol}\,\mathbb{H}^n$ for σ sufficiently nice. (i.e. Integrate over the Euclidean simplex Δ^n with the measure pulled-back from hyperbolic space $\mathbb{H}^n$.)

Remark 6.9. We have a pretty good idea of what this constant v_n should be in many dimensions.

1. $v_2 = \pi$ since all triangles have area $\leq \pi$ and an ideal regular triangle realizes area π.
2. For even dimensions, $v_{2k} = r\pi^k$ for some $r \in \mathbb{Q}$.
3. $v_n < \infty$ for all $n \geq 2$.
4. This supremum is always realized by ideal regular simplexes in each dimension.
5. $v_3 \approx 1.1\ldots$ so in dimension 3, Gromov norm is very close to volume which is very useful.

Let ω be a volume form on M so that $[\omega] \in H^n(M; \mathbb{R})$. We consider only straight chains because of the inequality of Gromov norm with the straightening map.

Exercise: Think through the inequality directions here.

Given any straight chain with $[\sum c_i\sigma_i] = [M]$, we calculate:

$$\begin{aligned}
\mathrm{Vol}(M) = \int_M \omega &= \langle [\omega], [M] \rangle \\
&= \langle [\omega], \left[\sum c_i\sigma_i\right] \rangle \\
&= \sum c_i \int_{\Delta^n} \sigma_i^* \omega \\
&\leq \sum |c_i| v_n && \text{(by definition of } v_n) \\
&\leq v_n \left\| \sum c_i \sigma_i \right\| = v_n \|M\|
\end{aligned}$$

□

As an example, consider a genus g surface. When we apply this inequality to the volume and Gromov norm, we have

$$\|\Sigma_g\| \geq \frac{1}{\pi} \mathrm{Area}(C_g)$$

where C_g is a genus g hyperbolic surface. Note that the Σ_g on the left is only a topological surface since we can define Gromov norm purely topologically. Then, by Gauss-Bonnet, we know that $\mathrm{Area}(C_g) = 2\pi|\chi(\Sigma_g)| = 4\pi(g-1)$.

Exercise: Use the above to prove that $\|\Sigma_g\| = 4g - 4$. Proceed as follows:

1. Triangulate Σ_g with $4g - 2$ geodesic triangles.
2. Conclude that $4g - 2 \geq \|\Sigma_g\|$.
3. General fact: for degree $0 \leq d < \infty$ and a covering map of degree d, $f : M \to N$, we have $\|M\| = d\|N\|$.
4. Take covers $\Sigma_h \to \Sigma_g$ and use 1 and 2 to argue that $\|\Sigma_g\| = 4g - 4$.

Remark 6.10. One might ask for the Gromov norm whether the inf over the norms is realized by some chain? In general, this answer is "no". Which leads to a follow-up question: what kind of mathematical object would realize the inf?

Exercise: We have a norm $\|\cdot\| : C_i(X) \to [0, \infty)$. Can we find some other space $C_i \subset Y$ where the limit of the sequence for a given Gromov norm is realized as some geometric object?

7 Lecture 7: Mostow Rigidity II

7.1 Finishing Gromov's Volume Theorem

Recall, we are proving Gromov's Volume Theorem 6.7. Last time, we showed that for M closed, orientable n-manifold, $\|M\| \geq \frac{\text{Vol}(M)}{v_n}$. Today, we prove the other inequality.

Proof of $\leq$ for Theorem 6.7. Given $\varepsilon > 0$, we show that there exists an n-cycle $[M] = [\sum a_i \sigma_i]$ with all a_i positive such that $\text{Vol}(\sigma_i) \geq v_n - \varepsilon$ for each i.

Let $M = \mathbb{H}^n / \Gamma$ and let Ω be a fundamental domain so that Ω is compact and $\Gamma \cdot \Omega = \mathbb{H}^n$. We continue the proof with $n = 3$ for simplicity.

Fix a point $x \in \Omega$. Given $g_1, g_2, g_3, g_4 \in \Gamma$, let $\Delta(g_1, g_2, g_3, g_4)$ be the straight simplex with vertices $g_1(x), g_2(x), g_3(x), g_4(x)$. Fix some large $L >> 1$. Then we can pick g_1, g_2, g_3, g_4 so that $\Delta(g_1, g_2, g_3, g_4)$ has side lengths between $[L - 2\,\text{diam}(\Omega), L + 2\,\text{diam}(\Omega)]$ since Γ acts transitively on the collection of fundamental domains. We define the following.

$$\mathcal{R}_L := \{\text{regular 3-simplices in } \mathbb{H}^3 \text{ of side length } L\}$$

$$\mu := \text{any Isom}^+(\mathbb{R}^n)\text{-invariant measure on } \mathcal{R}_L$$

$$\mathcal{S}(g_1, g_2, g_3, g_4) := \{\text{3-simplices in } \mathcal{R}_L \text{ with vertices lying in } g_1\Omega, g_2\Omega, g_3\Omega, g_4\Omega\}$$

Note that $\mathcal{S}(g_1, g_2, g_3, g_4)$ will be empty for most choices of g_i. So for most choices, $\mu(\mathcal{S}(g_1, g_2, g_3, g_4)) = 0$. Then we construct a chain

$$\hat{C}_L := \sum_{g_1, g_2, g_3 g_4} \pm \mu(\mathcal{S}(g_1, g_2, g_3, g_4)) \Delta(g_1, g_2, g_3, g_4)$$

where $\pm$ is given by the parity of the permutation of g_1, g_2, g_3, g_4. That is, whether $\sigma : \Delta^3 \to \Delta(g_1, g_2, g_3, g_4)$ is an orientation preserving or reversing map. This chain $\hat{C}_L$ is in $\mathbb{H}^3$ and we define $C_L := \pi(\hat{C}_L)$ to be a chain in M. If we fix $g_1 = \text{Id}$, then $\hat{C}_L$ is a finite chain. Note that if $L >> 0$, then the simplex is close to being a regular ideal simplex and so $\text{Vol}(\Delta(g_1, g_2, g_3, g_4)) \geq v_3 - \varepsilon$.

Claim 7.1. *C_L is a 3-cycle and $[C_L] = r[M]$ where r is the total measure on $\mathcal{R}_L$.*

Proof. Given a group, make a simplicial complex by $(n+1)$-tuples mod the diagonal action by a group.

A random triangle with side length L is the boundary of a random simplex with side lengths L in two different ways. The coefficients cancel out on either side, making the boundary 0.

I don't really know what these above comments mean. Maybe it would be worth trying to parse them at some point. □

□

7.2 Mostow Rigidity Theorem

Theorem 7.2 (Mostow Rigidity)**.** *Let M_1 and M_2 be closed hyperbolic n-manifolds with $n \geq 3$. If $\varphi : \pi_1(M_1) \to \pi_2(M_2)$ is an isomorphism of groups, then there exists an isometry $f : M_1 \to M_2$ such that $f_* = \varphi$ where f_* is the induced map on homotopy groups.*

Remark 7.3 (general spaces). Let's compare the statement of Mostow Rigidity with what is true for general topological spaces.

1. For general manifolds, π_1 doesn't determine homotopy type. So it is special that it is true for hyperbolic n-manifolds.
2. And in general, homotopy type doesn't determine homeomorphism type. (Example: lens spaces. There are lens spaces which are homotopy equivalent but not homeomorphic. See the wikipedia page.)
3. Recall: if M_1 and M_2 are aspherical CW complexes (2-dimensional?), then $M_1 \cong M_2$ if and only if $\pi_1 M_1 \cong \pi_1 M_2$.

We have a chain of implications for equivalence of topological spaces.

$$\text{isometry} \Rightarrow \text{diffeo} \Rightarrow \text{homeo} \Rightarrow \text{homotopy eq} \Rightarrow \pi_1 \text{ is isomorphic}$$

In general, we cannot go "back up" the chain, but Mostow rigidity allows us to go all the way back.

Remark 7.4 (local rigidity). Local rigidity is the idea that if you perturb the metric on a hyperbolic 3-manifold, then you cannot reach a non-isometric hyperbolic structure. Local rigidity was done before Mostow by Weil with algebraic geometry tools. We can actually prove local rigidity with the Schláffli formula.

1. Cut M into geodesic simplices.
2. Perturb the simplices, but the angle around the vertices must stay the same.
3. Apply the Schläffli formula to say that the volume stays the same. (Does not prove total invariant, just volume)

Exercise: Work out the proof above.

So how do we go from the invariant volume given by the Gromov formula to the isometry type as a homotopy invariant?

Outline of Mostow Rigidity proof. We give an outline of this proof. A good source for all the details is this master's thesis by A. Lücker.

1. Given the isomorphism of fundamental groups, there is a homotopy equivalence $f : M \to N$ such that $\varphi = f_*$. Then we lift to $\tilde{f} : \mathbb{H}^n \to \mathbb{H}^n$. Then $\tilde{f}$ is π_1-equivariant so that $\tilde{f}(g \cdot x) = \varphi(g) \cdot \tilde{f}(x)$ for all $g \in \pi_1 M$. **Goal:** Show that $\tilde{f}$ is equivariantly-homotopic to some isometry of $\mathbb{H}^n$.

2. $\tilde{f}$ is a (k, c)-quasi-isometry. Show that $\tilde{f}$ induces a map $\partial \tilde{f} : \partial \mathbb{H}^3 \to \partial \mathbb{H}^3$ on the sphere at ∞. And $\tilde{f}$ takes geodesics to quasi-geodesics.

3. $\partial \tilde{f}$ is a quasi-conformal map.

4. Use that $\mathrm{Isom}(\mathbb{H}^n) \cong \mathrm{Conf}(S^2)$. We can go from quasi-conformal to conformal because $\partial \tilde{f}$ respects the group action. Given this, we have that $\partial \tilde{f} = \partial \psi$ on the boundary for some isometry ψ of $\mathbb{H}^3$. Then there is some D such that $d(\tilde{f}(x), \psi(x)) \leq D$ for all $x \in \mathbb{H}^3$ and then we apply a straight-line homotopy.

We need to show that $\partial \tilde{f}$ is actually a conformal map on $\partial \mathbb{H}^3 = S^2_\infty$. This follows because $\tilde{f}$ sends 4-tuples in $\partial \mathbb{H}^3$ to 4-tuples. And the image of the vetices of a regular ideal simplex should still be a regular ideal simplex using Gromov's theorem.

Exercise: A homeomorphism of $\mathbb{H}^2$ taking vertices of regular triangles to vertices of regular triangles is conformal. prove or disprove.

Exercise: Read and work out the details of this proof.

□

8 Lecture 8: $\mathbb{H}^n/\Gamma$ and Dehn surgery

8.1 Examples of $\mathbb{H}^n/\Gamma$

Let's look at a few examples of hyperbolic 3-manifolds to start. Let $\Gamma < \mathrm{Isom}^+(\mathbb{H}^n)$ be a discrete subgroup with cofinite volume (i.e. $\mathbb{H}^n/\Gamma$ has finite volume). Let $\Gamma \circlearrowright \mathbb{H}^n$ freely and this happens iff Γ is torsion free.

Exercise: Prove that Γ is discrete if and only if it acts properly and discontinuously on $\mathbb{H}^n$.

Dimension 2.

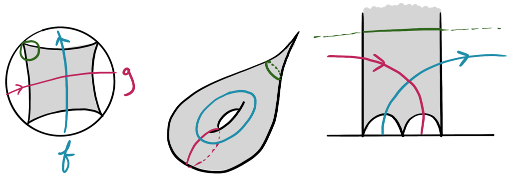

Figure 11: The punctured torus is a quotient of $\mathbb{H}^2$ by two hyperbolic translations.

1. $\dim M = 2$. Define $\Gamma_2(n) < \mathrm{SL}_2 \mathbb{Z}$ for $L \in \mathbb{N}$ to be the group of matrices in $\mathrm{SL}_2 \mathbb{Z}$ congruent to Id mod n.

$$\Gamma_2(n) = \{nA + \mathrm{Id} \mid A \in \mathrm{SL}_2 \mathbb{Z}\}$$

For $L \geq 3$, this generates a free group. And $\mathbb{H}^2/\Gamma_2(n)$ is a hyperbolic surface with finite volume.

2. $\Gamma = \langle f, g\rangle$ where f and g are hyperbolic translations in $\mathrm{PSL}(2, \mathbb{R})$ with different endpoints on the circle at ∞. And we require that $fgf^{-1}g^{-1}$ is a parabolic transformation (i.e. it has one fixed point on the boundary of the circle at ∞). Then $\mathbb{H}^2/\Gamma$ is $T^2 - \{\text{point}\}$. Up to conformal automorphism, the commutator $[f, g]$ is a parabolic element $z \mapsto z + r$. The *end* is homeomorphic to $S^1 \times [0, \infty)$ and has metric $dt^2 + e^{-t}dx^2$. Note that $\pi_1(T^2 - \{\text{pnt}\})$ is F_2 and so the group $\Gamma = \langle f, g\rangle$ is free. So we have a representation $\rho : \pi_1(T^2 - \{\text{pnt}\}) \to \mathrm{PSL}(2, \mathbb{R})$ and every element of Γ is of hyperbolic type or conjugate into $\langle [f, g]\rangle$. See Figure 11.

Dimension 3.

3. Consider $\mathrm{PSL}_2 \mathbb{Z}[i] < \mathrm{PSL}(2, \mathbb{C})$. This is not a torsion-free subgroup, but we want to consider subgroups of $\mathrm{PSL}_2 \mathbb{Z}[i]$ which are torsion-free. Let Λ be the group generated by translations $z \mapsto z + a + bi$ for $a, b \in \mathbb{Z}$. Note that the orbit of $(1, 0)$ under $\mathrm{SL}_2 \mathbb{Z}$ is $\mathbb{Q}$. I remember getting really lost in this example and I wasn't sure what Benson was saying by the end.

4. Hyperbolic knot complements. For a knot K, we define $M_K := S^3 \setminus \mathrm{nbhd}(K)$. This is a compact 3-manifold with $\partial M_K \cong T^2$.

Theorem 8.1 (Thurston)**.** *If K is not a torus knot or a satellite knot*[9]*, then the interior of M_K has a complete hyperbolic metric of finite volume.*

[9] You can learn about satellite knots here and torus knots here.

Note in that the case where K is a satellite knot, the complement $S^3 \setminus K$ has an essential tours, so it cannot have a hyperbolic metric by Theorem 3.1.

In light of Theorem 8.1, when K is not torus or satellite, $M_K \cong \mathbb{H}^3/\Gamma$ and $\Gamma \cong \pi_1(S^3 \setminus K)$ for $\Gamma < \mathrm{PSL}(2, \mathbb{C})$ by Mostow Rigidity. In fact, this Γ actually lies in $\mathrm{PSL}_2\, k$ for a number field k in $\mathbb{C}$.

Exercise: Trefoil Knot fundamental group: $1 \to \mathbb{Z} \to \langle a, b \mid aba = bab \rangle \to \mathrm{SL}_2\, \mathbb{Z} \to 1$ and the middle term (braid group) is $\pi_1(S^3 \setminus \text{trefoil})$ Another geometry on the trefoil complement.

I do not remember what the exercise was. Maybe to work out what Γ will be in this case?

8.2 Dehn Surgery on a knot

If $\partial M_K \cong T^2$, then we can perform a dehn filling to obtain a new 3-manifold without boundary $M_K(p, q)$. If γ is the knot, then $M_K(p, q) = M_K \sqcup (D^2 \times \gamma)$ where we identify the boundaries by gluing map $\begin{pmatrix} p & r \\ q & s \end{pmatrix} \in \mathrm{SL}_2\, \mathbb{Z}$. Note that r, s are uniquely determined by p, q because of the Euclidean algorithm. In particular, $M_K(1, 0) \cong S^3$ because we have just reversed the gluing process. Then we can look at what this does to fundamental groups. Let a, b be the generators of $\mathbb{Z}^2 = \pi_1(\partial M)$ inside $\pi_1(M_K)$. Then we have the following relation.

$$\pi_1(M_K(p, q)) = \pi_1(M_K)/\langle a^p b^q = 1 \rangle$$

Exercise: work out this construction.

Remark 8.2. It turns out that every closed, connected, oriented 3-manifold can be realized by performing Dehn surgery on a link in S^3. This is known as the Lickorish-Wallace Theorem.

Theorem 8.3 (Thurston, Hyperbolic Dehn Surgery Theorem). *Let M_K be hyperbolic. Then for all $(p, q) \in Z^2$ with $\gcd(p, q) = 1$ except for finitely many pairs, $M_K(p, q)$ is a closed hyperbolic 3-manifold.*

The figure 8 knot has the most "bad pairs" at 10. The $(-2, 3, 7)$-pretzel knot has 7 bad pairs, and other than that all other knots have at most 6 bad pairs.

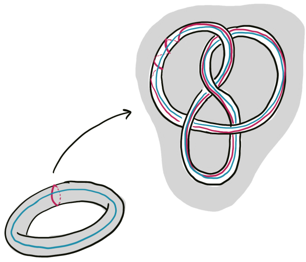

Figure 12: Dehn surgery on the complement of the figure 8 knot. Glue in the solid torus by identifying the blue curves and the pink curves.

8.3 Hyperbolic volumes of 3-manifolds

Define a set $\mathcal{V} \subset \mathbb{R}^+$ to be all volumes of hyperbolic 3-manifolds.

$$\mathcal{V} = \{\mathrm{Vol}(M^3) : M^3 \text{ is complete, finite volume, hyperbolic}\} = \{\mathrm{Vol}(\mathbb{H}^3/\Gamma) : \Gamma < \mathrm{PSL}(2, \mathbb{C})\}$$

Theorem 8.4 (Thurston 1978)**.** –

1. *For all* $D > 0$*, then set* $\{M : \mathrm{Vol}(M) = D\}$ *is finite. So the map* Vol : $\{$*hyperbolic, finite volume* $M^3\} \to \mathbb{R}$ *is finite-to-one.*

2. *The subset* $\mathcal{V} \subset \mathbb{R}^+$ *is closed, non-discrete, and well-ordered ordinal type* ω^ω*.*

Remark 8.5. It is still an open question whether there are M_1 and M_2 with $\frac{\mathrm{Vol}(M_1)}{\mathrm{Vol}(M_2)}$ an irrational number.

Volumes go down with Dehn surgery. There is a reason for this later on in the notes. To prove that the set is well-ordered, there is an inductive argument using this fact about Dehn surgery.

Teaser for next time:

For a Riemannian manifold M, we define the *injectivity radius* for a point.

$$\mathrm{InjRad}(x) := \sup\{r : B_r(x) \subset M \text{ is embedded}\}$$

Pick some $\varepsilon > 0$ and then define $M_{[0,\varepsilon)} = \{x \in M : \mathrm{InjRad}(x) \leq \varepsilon\}$; this is the *thin part* of the manifold. The *thick part* is $M \setminus M_{[0,\varepsilon)}$. We find that if we make ε small enough, the thin part will contain none of the interesting topology of the manifold.

Suppose we have a space X with $X = \bigcup U_i$ and each U_i open and contractible and $U_{i_1} \cap \cdots \cap U_{i_k}$ either empty or contractible for every subset of indices. Then $X \cong \mathit{nerve}(U_i)$. We will apply this method to the thick/thin decomposition to figure out the topology of X.

9 Lecture 9: Geometry of Hyperbolic 3-Manifolds

9.1 Main topics moving forward

First, we look at three main topics which we will address in the coming lectures.

1. **Thick-Thin Decomposition**: *"A canonical length scale in hyperbolic n-manifolds."* look at Benedetti to address cases of higher dimensions.

   Lemma 9.1 (Thick-Thin Decomposition)**.** *For each n, there exists a universal constant $\varepsilon_n > 0$ such that if M^n is a complete hyperbolic finite-volume n-manifold, and $M_{[0,\varepsilon_n]}$ is the subset of M with injectivity radius $\leq \varepsilon_n$, then this subset has a very specific form. It is a union of*

   (i) tubular neighborhoods of an embedded closed geodesic of length $\leq 2\varepsilon_n$;

   (i) a neighborhood of a cusp.

   At the end of the last lecture, we discussed making $M_{[0,\epsilon)}$ "thin enough" so it doesn't contain any of the interesting topology of the manifold. The Thick-Thin Decomposition Lemma tells us that we can find a small enough ε for this and it only depends on the dimension.

2. **Thurston's hyperbolic Dehn surgery theorem**

   Definition 9.2 (Dehn Surgery)**.** Let M be a 3-manifold with $\partial M = T$ a torus. Choose μ and λ to be generators for $H_1(\partial M)$ and represented by simple closed curves. Let $\frac{p}{q} \in \mathbb{Q} \cup \{\infty\}$. Then we define $M_{p/q}$ to be

   $$M_{p/q} := M \sqcup_T (\mathbb{D} \times S^1)$$

   where the meridian (non-trivial curve) of the solid torus is glued along the curve $p\mu + q\lambda$ on T.

   Now let M be a finite volume complete hyperbolic 3-manifold with $n > 0$ cusps which are all tori. Then for all but finitely many choices of slope on each boundary component, the result of Dehn filling M along these slopes is hyperbolic. This is Theorem 8.3.

   Remark 9.3. You might wonder if a non-closed finite-volume hyperbolic 3-manifold always has torus cusps/boundary. The answer is yes. The cusps are the quotient of a horosphere by an isometry group. A horosphere in $\mathbb{H}^3$ has a Euclidean structure (think of a plane parallel to the boundary in the upper half-plane model). Hence, the cusps are either T^2 or the Klein bottle, but must be orientable, and thus a torus.

3. **Volumes**

$$\mathcal{V} := \{\text{volumes of complete hyperbolic 3-manifolds}\}$$

Then $\mathcal{V}$ is a closed subset of $\mathbb{R}$, well-ordered, of type ω^{ω}[10]. And, delightfully, the map

$$\{\text{hyperbolic 3-manifolds/isometry}\} \to \mathcal{V}$$

is a finite-to-one map. If v is a limit ordinal of type k, then Vol_v is the volume of at least one M with exactly k cusps[11].

This means that if M' is obtained by Dehn filling the cusps of M, then $\mathrm{Vol}(M') < \mathrm{Vol}(M)$ and as these fillings $\to \infty$, $\mathrm{Vol}\, M' \to \mathrm{Vol}\, M$.

Look at the representations $\rho : \pi_1 M \to \mathrm{PSL}(2, \mathbb{C})$ up to conjugacy and this is a character variety. There is one point corresponding to a unique hyperbolic structure with other points accumulating.

Exercise: Figure this out What is this saying?

9.2 Thick-Thin Decomposition

Thick-Thin Decomposition will follow from the following algebraic lemma. We will apply this geometrically by looking at short closed geodesics which we use to cut up our surface. In the following, the paths from p to $g \cdot p$ will correspond to these short closed geodesics in the quotient.

Definition 9.4. A group is *virtually nilpotent* if it has a subgroup of finite index which is nilpotent. A group is k-step nilpotent if for all $a_0, a_1, \ldots, a_k$, their nested commutator is trivial. That is,

$$[\ldots[[a_0, a_1], a_2]\ldots] = 1.$$

Lemma 9.5 (Margulis)**.** *Let Γ be a discrete group of isometries of $\mathbb{H}^n$. Then there is $\varepsilon_n > 0$ such that*

$$\Gamma(p, \varepsilon_n) := \langle g \in \Gamma \mid d(p, g \cdot p) < \varepsilon_n \rangle$$

is virtually nilpotent.

---

[10] If you don't *really* know what this means, you are in good company.

[11] This "type k" has to do with how many "levels down" it is in the list of orbitals. Again, I don't really know what this means.

Remark 9.6. I asked Danny how I should think of nilpotent geometrically and he cryptically said that I should think of “iterated circle bundles over a torus”. I do not know what this means or how to think about this.

proof of lemma. Let $G = \mathrm{Isom}^+(\mathbb{H}^n)$ and let $\Gamma < G$. Then fix a left-invariant metric on G and define

$$\Gamma(\varepsilon) := \langle g \in \Gamma \mid d(g, \mathrm{Id}) < \varepsilon \rangle .$$

Fix a point $p \in \mathbb{H}^n$ and define $\Gamma(p, \varepsilon)$ in the same way as in the lemma above.

$$\Gamma(p, \varepsilon) := \langle g \in \Gamma \mid d(p, g \cdot p) < \varepsilon \rangle$$

For the appropriate metric on G, we have that $\Gamma(\varepsilon) < \Gamma(p, \varepsilon)$. This follows because if an isometry is very close to the identity then it can only move points a little bit. However, you can have an element which moves the point p only a little bit but is far from the identity (e.g. if g is in the stabilizer of p). We identify $\mathrm{stab}(p)$ with $O(n)$ in a canonical way.

Claim 9.7. *There is some $\varepsilon > 0$ such that $\Gamma(\varepsilon)$ is nilpotent.*

Proof. We examine the commutator map.

$$[-,-] : G \times G \to G \quad \text{and note that} \quad \mathrm{Id} \times \mathrm{Id} \mapsto \mathrm{Id} .$$

Then consider the differential at the point $\mathrm{Id} \times \mathrm{Id}$.

$$d[-,-] : T_{\mathrm{Id}\times\mathrm{Id}}(G \times G) = T_{\mathrm{Id}}G \times T_{\mathrm{Id}}G \to T_{\mathrm{Id}}G$$

We look at a spanning set of the tangent space given by $T_{\mathrm{Id}}G \times 1$ and $1 \times T_{\mathrm{Id}}G$ and then we see that $d[-,-] = 0$ for both of these[12]. Hence, we have $d[-,-] = 0$ at the identity. This means that $[-,-]$ is contracting at $\mathrm{Id} \times \mathrm{Id}$. And if we iterate the map then points will be attracted to Id.

So consider the set

$$S_\varepsilon := \{g \in \Gamma \mid d(g, \mathrm{Id}) < \varepsilon\}.$$

This set will be finite because Γ is discrete. Also, if we take iterated commutators of elements in S_ε, then the group elements will approach Id based on what we discussed above. All these elements will be in Γ. But again since Γ is discrete, we know that there is a k depending on Γ such that for all $s_0, \ldots, s_k \in S_\varepsilon$,

$$[\ldots[s_0, s_1], s_2], \ldots s_k] = \mathrm{Id} .$$

Hence, $\Gamma(\varepsilon) = \langle S_\varepsilon \rangle$ is nilpotent. This proves the claim. □

[12]To see this, choose any path in G based at Id given by $g(t)$ such that $g(0) = \mathrm{Id}$. Then consider $\frac{d}{dt}|_{t=0}[g(t), \mathrm{Id}]$ and note that this derivative will be 0.

Finally, one can show that $\Gamma(\varepsilon)$ is finite index inside $\Gamma(p, \varepsilon)$. This follows from the compactness of $O(n)$. The idea is to take a finite number of translates of $O(n)$ which cover $\Gamma(p, \varepsilon)$ and then use finite index to get virtually nilpotent. $\square$

Now we explain how the Margulis Lemma implies Lemma 9.1.

Claim 9.8. *Lemma 9.5 implies Lemma 9.1.*

Proof. Take $\Gamma = \pi_1 M$ for a complete hyperbolic manifold. i.e. $\Gamma < \mathrm{Isom}^+ \mathbb{H}^n$ torsion-free and discrete. Let ε_n be the constant given by the Margulis Lemma 9.5. We consider what the subgroups $\Gamma(\varepsilon_n, p)$ can look like. By the Margulis Lemma, each of these subgroups should be virtually nilpotent.

Suppose that $g \in \Gamma$ is a hyperbolic element. If h is also hyperbolic, then $\langle h, g \rangle$ contains a free group. Note that F_n for $n \geq 2$ is *not* virtually nilpotent. This means that:

(i) if $\Gamma(\varepsilon_n, p)$ contains hyperbolic g, then $\Gamma(\varepsilon_n, p) \cong \mathbb{Z}$ and is generated by a root of g;

(ii) if $\Gamma(\varepsilon_n, p)$ contains a parabolic g, then everything else in the group is parabolic with the same fixed point. If we started with a finite-volume manifold, then in this case $\Gamma(\varepsilon_n, p) \cong Z^n$.

Then we think through what $M = \mathbb{H}^n/\Gamma$ looks like around each of these subgroups. We see that *(i)* in the above list corresponds to the tuber neighborhoods of short closed geodesics in Lemma 9.1 and *(ii)* in the above list gives the neighborhood of a cusp in Lemma 9.1. $\square$

Fix a volume v and look at all hyperbolic manifolds with volume $< v$. Then the thick part $M_{[\varepsilon_n,\infty)}$ also has volume $< v$. There are only finitely many topological options for what the thick part can be. Start with a thick part $M_{[\varepsilon_n,\infty)}$ of a general manifold and apply the following process.

1. Take points and make a voroni tessellation.
2. The number of points we need is bounded by the lower bound on the injectivity radius and upper bound on the volume.
3. So we can decompose the thick part into finitely many polyhedra glued in finitely many ways.

Note that $\pi_1 M = \pi_1 M_{[\varepsilon_n,\infty)}$. This is because adding the neighborhood of a cusp or a solid torus (neighborhood of a s.c.c.) does not change the π_1. Hence, the π_1 and (by Mostow rigidity) the isometry type is determined only by the thick part.

Remark 9.9 (Cute generalization). Let $\mathcal{V}$ be the set of finite volumes of complete 3-manifolds.

- M hyperbolic $\mapsto \mathrm{Vol}(M)$
- M euclidean $\mapsto 0$
- M spherical $\mapsto -\mathrm{Vol}(M)$

Then $\mathcal{V} \subset \mathbb{R}$ is closed and well-ordered of type ω^ω and $\{\text{manifolds}\} \to \mathcal{V}$ is a finite-to-one map.

10 Lecture 10: Thurston's Hyperbolic Dehn Surgery Theorem

Let M be a compact, oriented 3-manifold with $\partial M = T^2$ whose interior admits a complete hyperbolic structure of finite volume. Pick μ and λ to be meridian and longitude curves in the boundary T^2. Let $M_{p/q}$ be the p/q Dehn filling of M.

Theorem 10.1 (Thurston Hyperbolic Dehn Surgery, (full statement))**.** *For almost all p and q, $M_{p/q}$ defined above admits a hyperbolic structure. Furthermore, the "core" of the added solid torus c becomes a geodesic in $M_{p/q}$ and* $\mathrm{len}(c) \to 0$ *as* $(p, q) \to \infty$*.*

Also the geometry of the thick part of $M_{p/q}$ approaches the geometry of the thick part of M[13]*. From this it follows that* $\mathrm{Vol}(M_{p/q}) \to \mathrm{Vol}(M)$ *and an argument on Gromov norm implies that* $\mathrm{Vol}(M_{p/q}) < \mathrm{Vol}(M)$ *for all* $p/q \in \mathbb{Q} \cup \{\infty\}$*.*

Note that the boundary torus is a quotient of a horoball so it has a well-defined Euclidean metric up to scaling. Thus, it makes sense to talk about the "length" of curves on the boundary. This is relevant because $(p, q) \to \infty$ if the length of the geodesic $\to \infty$. This happens if and only if either $p \to \infty$ or $q \to \infty$.

Remark 10.2. In full generality, there is a statement of this theorem for any finite number of cusps. In that case, you would be doing Dehn fillings on a link with more than one component.

[13] In the sense of Gromov-Hausdorff convergence.

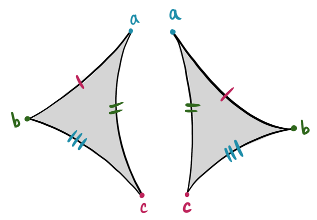

Figure 13: Gluing two triangles with these identifications makes a thrice-punctured sphere.

10.1 Gluing ideal triangles

Remark 10.3. For the next two sections, a good external reference is *Hyperbolic Knot Theory* by Jessica Purcell [1].

Eventually, the goal is to think about gluing together simplices to get hyperbolic structures on 3-manifolds. In particular, we will want to use *ideal* simplices in order to get a 3-manifold with a cusp as in the theorem. But to start with, we will glue together ideal triangles to get hyperbolic structures on surfaces.

Example 10.4 (Thrice-punctured sphere). First consider gluing two ideal triangles and match up the edges directly to form a thrice-punctured sphere. Recall that all ideal triangles are isometric by basic hyperbolic geometry/complex analysis. *And* there is only one way to match them up face to face.

Question 3. *Does this mean there is only one hyperbolic structure on* $S_{0,3}$*?*

The answer is no! There is actually a 3-dimensional parameter space of hyperbolic structures! How do we see this?

For Euclidean triangles, gluing two edges by an isometry is unique once you match up the vertices because the edges have finite length. But ideal hyperbolic triangles have sides with infinte length. We want to identify $\mathbb{R}$ with $\mathbb{R}$ with an isometry: we can do this with any translation of $\mathbb{R}$. There are three such edge identifications, and thus three shear parameters, and the space of hyperbolic structures is $\mathbb{R}^3$. Let $s_{ab} \in \mathbb{R}$ represent the shear parameter along the edge connecting vertices a and b.

Question 4. *Which hyperbolic structures are complete?*

Heuristically, a complete hyperbolic structure on $S_{0,3}$ is one in which the puncture is a cusp and infinitely far away from the rest of the surface. A non-complete structure has a "missing boundary component". See Figure **??** for what this looks like.

For a hyperbolic structure on $S_{0,3}$, recall that we have a holonomy representation

$$\rho : \pi_1(S_{0,3}) \to \mathrm{Isom}^+(\mathbb{H}^2)$$

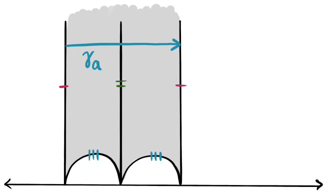

Figure 14: A picture of two triangles in the universal cover with the lifts of a, γ_a, and other sides labeled. The translation across two triangles is indicated with an arrow.

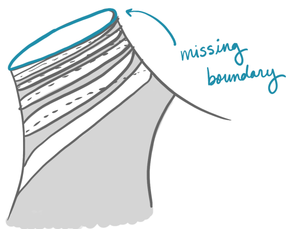

Figure 15: Around a “missing geodesics” the triangles spiral around and limit to a circle instead of a point. This gives an *incomplete* hyperbolid structure.

in the sense of developing maps[14]. Consider a puncture a of $S_{0,3}$ and let γ_a be a loop encircling the puncture. The hyperbolic structure is complete at a if $\rho(\gamma_a)$ is a parabolic element in $\mathrm{PSL}(2,\mathbb{R}) \cong \mathrm{Isom}^+(\mathbb{H}^2)$. Geometrically, when we lift the curve γ_a to $\mathbb{H}^2$ (so that ∞ is a pre-image of a), it should be a horizontal line and the holonomy should be translation by two triangles. This means that the total shear across both edge identifications involving a should be 0.

$$s_{ab} + s_{ac} = 0$$

If this condition is not satisfied, then there is a “missing geodesic” on the boundary of $S_{0,3}$ and the length of this missing geodesic is $s_{ab} + s_{ac}$. See Figure 15.

For the entire structure to be complete, this needs to be true around each vertex. So we get three linear relationships on $\mathbb{R}^3$, giving a dimension 0 subset which is satisfied exactly when all the shifts are 0. Hence, there is a unique *complete* hyperbolic structure on $S_{0,3}$.

[14]For a thorough treatment of this, see Chapter 3 of Purcell’s “Hyperbolic Knot Theory”.

Exercise: Work through the details of this:

1. Why is this notion of completeness the same as "geodesically complete" from Riemannian geometry?
2. Why is the length of the missing geodesic as given?
3. How does the "degenerate tessellation" picture of the universal cover relate to the hyperbolic structure being incomplete?

Example 10.5 (Once-punctured torus). In the second example, we identify the two triangles to make a torus with a puncture. First glue to make a rectangle, then glue to make a punctured torus. In this case, we still have three shift parameters so the space of all hyperbolic structures is $\mathbb{R}^3$.

Again, we ask which hyperbolic structures are complete. Notice that the link of the single puncture contains a neighborhood of all six vertices of the two triangles. For the structure to be complete we need the holonomy around this vertex to be parabolic, and thus the total shift over these sides needs to be 0. So we have one condition

$$s_{ac} + s_{ab} + s_{bc} + s_{ac} + s_{ab} + s_{bc} = 0$$

I think I may be describing the shifts incorrectly maybe there are some extra signs thrown in here. In any case, this cuts out a 2-dimensional submanifold of $\mathbb{R}^3$ of complete hyperbolic structures. In fact, this is actually an algebraic variety! It gives another way parameterize the Teichmüller space of the once-punctured torus.

10.2 Ideal simplices in $\mathbb{H}^3$

Every ideal simplex in $\mathbb{H}^3$ can be characterized by a *simplex parameter* which is a complex number $z \in \mathbb{H}^2$ up to isometry. We can always position 3 vertices of the tetrahedron on $0, 1, \infty$ up to isometry of $\mathbb{H}^3$ (i.e. action by $\mathrm{PSL}(2, \mathbb{C})$). The last vertex will land somewhere in $\partial\mathbb{H}^3 \cong \hat{\mathbb{C}}$ and let this number be called z. Up to another isometry of $\mathbb{H}^3$, we can assume that $\mathrm{Im}(z) > 0$ and so we can take $z \in \mathbb{H}^2$ modeled by the upper half plane. See Figure 17 for what this simplex will look like in $\mathbb{H}^3$.

What are the isometries of an ideal simplex? First, S_4 acts by permuting the vertices. But if we want orientation preserving isometries, we take $A_4 < S_4$. There is a subgroup $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ which permutes two non-adjacent edges.

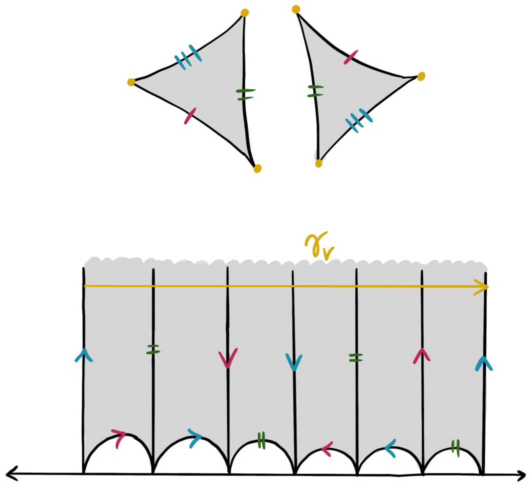

Figure 16: Gluing two triangles with these edge identifications makes a punctured torus. Then we show a tessellation in the universal cover.

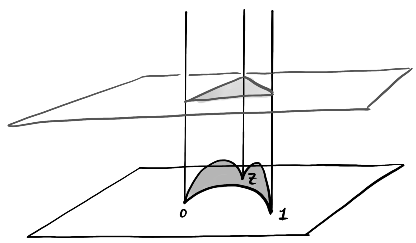

Figure 17: An ideal simplex in $\mathbb{H}^3$ with simplex parameter z.

We want to define an isometry of $\mathbb{H}^3$ which does this permutation on the vertices of our simplex with vertices at $0, 1, \infty, z$. Pick an isometry of $\mathbb{H}^3$ (i.e. a Möbius transformation) so that $\infty \leftrightarrow 0$ and $z \mapsto 1$. Then we must have $1 \mapsto z$ because of the way the cross-ratio is defined.

Exercise: Check that $1 \mapsto z$.

Lastly, note that the Euclidean triangles $T(0, 1, z)$, $T(0, 1, \frac{z}{z-1})$ and $T(0, 1, \frac{1}{1-z})$ are all similar. Hence, if you complete each of these with an ideal simplex by putting the additional vertex at ∞, then they differ by a Möbius Transformation (a Euclidean dilation).

So all ideal simplices in $\mathbb{H}^3$ are parameterized by z with $\text{Im}(z) > 0$ up to the ambiguity $z' = \frac{z}{z-1}$ and $z'' = \frac{1}{1-z}$.

10.3 Gluing together ideal simplices

Let M^3 be a hyperbolic 3-manifold with $\partial M^3 \cong T^2$. Suppose that X^3 is a cell complex with 1 vertex v such that $X^3 - N(v) \cong M^3$. We can try to put a hyperbolic structure on $\text{int}(M^3)$ by choosing simplex parameters $z_1, \ldots, z_t$ associated to the simplices of X and then figure out equations that must be satisfied by the z_i to get a hyperbolic structure on $\text{int}(M^3)$.

Example 10.6. Take four simplices glued together in the picture. The link of the vertex v is the four triangles (cross-sections of simplices) glued together. They are parallel to the opposite faces of the simplices from v, so we will think about gluing these faces together instead. Since these are hyperbolic ideal simplices, we have a well-defined hyperbolic structure on the interior of the simplices and on the 2-faces. So we need to check that all the edges work out.

We need that *the angle around each edge of the gluing has total angle* 2π. Going around an edge gives an isometry from the edge to itself. This should be the identity.

Let z_1, z_2, z_3, and z_4 be the simplex parameters of the four simplices glued around edge e. We will think of laying down the triangles (the "bases" of the simplices) in the complex plane. Put the endpoint of e at 0 and the first outer vertex at 1. Then the other vertices will be given by z_1, z_1z_4, $z_1z_4z_3$, and $z_1z_4z_3z_2$. See Figure 18 to see the neighborhood of an edge. In order for the start and end vertices to match up, we need that

$$1 = z_1z_4z_3z_2.$$

By going around each of the edges in the simplex gluing, we get some equation of this form. These are called the *edge equations*. They are useful because they give an algebraic condition for when we have a hyperbolic structure, but they are not sufficient

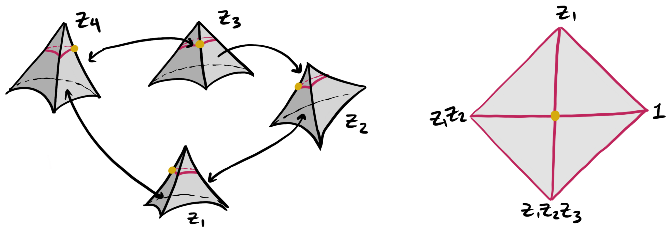

Figure 18: Four simplices are glued around an edge in a triangulation. The simplex parameters determine the edge equations so that $1 = z_1 z_2 z_3 z_4$.

to have a hyperbolic structure. They only guarantee that the angle around an edge is a multiple of 2π. Also, the edge equations give us a representation $\pi_1 M \to \text{Isom}^+(\mathbb{H}^n)$. I have no idea about this last sentence.

To get *complete* characterization for when we have a *complete* hyperbolic structure, we also need that

$$\sum \log(z_i) = 2\pi i.$$

In contrast, this condition is "combinatorial" in the sense that it doesn't cut out an algebraic variety. But it is locally constant on the variety that we do have, since the sum must be an integer multiple of 2π.

Exercise: What is this $\log$ condition saying geometrically and why is the sum always a multiple of 2π?

11 Lecture 11: Proof of Hyperbolic Dehn Surgery Theorem

To get a good picture of a geometry on a hyperbolic 3-manifold and how it can change with Dehn fillings, we can use an ideal triangulation. We will then look at the space of representations of $\pi_1 M$ into $\text{PSL}(2,\mathbb{C})$ and find a "lower bound on representations". This will give a hyperbolic structure on the desired manifold M.

Recall the statement of Thurston's Hyperbolic Dehn Surgery Theorem 10.1. In this lecture, we will give a proof of this statement.

Remark 11.1 (Probably something Benson said). The algebraic perspective on this result is the following. We have a discrete faithful representation $\rho : \Gamma \to \text{Isom}^+(\mathbb{H}^3) = \text{PSL}(2,\mathbb{C})$ with $\Gamma = \pi_1 M$ such that $\text{Vol}(\mathbb{H}^3/\Gamma) < \infty$. Then suppose that Γ is not co-compact. Then it contains a $\mathbb{Z}^2$ subgroup generated by two parabolic elements with the same fixed point: call them a and b. For all p, q we have a new subgroup $\Gamma_{p,q} := \Gamma/\langle a^p b^q \rangle$. Thurston's theorem says that for all except finitely many p, q, we have that $\Gamma_{p,q} \hookrightarrow \mathrm{PSL}(2, \mathbb{C})$ is discrete, faithful, and cocompact.

11.1 Main ingredients of the proof

Let M be a compact, oriented, hyperbolic 3-manifold with $\partial M = T^2$ as in Theorem 10.1. Then let $X(M)$ be the *character variety of* M defined by $\mathrm{Hom}(\pi_1 M, \mathrm{PSL}(2,\mathbb{C}))//\text{conjugacy}$. In a neighborhood of a complete structure, $X(M)$ is simply $\mathrm{Hom}(\pi_1 M, \mathrm{PSL}(2,\mathbb{C}))/\text{conjugacy}$ and so if you are unfamiliar with GIT quotients (the double slash //), don't worry about it. A point in $X(M)$ representing the complete structure corresponds to a discrete, faithful representation and we call it the *geometric point*.

Fact 1: The dimension $\dim_{\mathbb{C}} X(M)$ near the geometric point is equal to the number of cusps of M. Most of the nearby representations are not discrete. Those which are discrete are not also faithful; in this case, the representation is a quotient of $\pi_1 M$ (i.e. it is $\pi_1 M$ with some extra relations to add torsion).

Fact 2: Let $\rho_0 : \pi_1 M \to \mathrm{PSL}(2,\mathbb{C})$ be the discrete, faithful rep (corresponding with the geometric point). Let $\rho : \pi_1 M \to \mathrm{PSL}(2,\mathbb{C})$ be a nearby representation in $X(M)$. Then $\rho|_{\pi_1(\partial M)}$ is conjugate into $\mathbb{C} \rtimes \mathbb{C}^*$ (which is the subgroup of $\mathrm{Isom}^+(\mathbb{H}^3)$ fixing ∞). Recall that $\pi_1(\partial M) \cong \mathbb{Z}^2 = \langle m, l \mid m l m^{-1} l^{-1} \rangle$. We can represent the elements in $\mathbb{C} \rtimes \mathbb{C}^*$ by matrices of the form

$$\begin{bmatrix} \mu & \nu \\ 0 & \mu^{-1} \end{bmatrix}.$$

For such an element, μ is the translation factor and ν is the dilation factor. Let $h_\rho(m)$ and $h_\rho(l)$ in $\mathbb{C}^*$ be the dilation factors of $\rho(m)$ and $\rho(l)$.

Exercise: Work out why we can conjugate $\pi_1(\partial M)$ into this subgroup. It follows because the two generators commute so they must have the same fixed points. And they cannot both be hyperbolic elements. So we wind up with two parabolic elements with the same fixed point.

Fact 3: If we have a complete hyperbolic structure, then $\pi_1(\partial M) \to \pi_1 M$ is injective near ρ_0. We can conjugate so that

$$\rho_0(m) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \rho_0(l) = \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$$

Then, near ρ_0, we have $\frac{\log h_\rho(l)}{\log h_\rho(m)} = c$.

If we deform $\rho_0(m)$ and $\rho_0(l)$ a little bit, they become non-identity hyperbolic elements. Since $\rho_0(m)$ and $\rho_0(l)$ are linearly independent, for ρ near ρ_0, there are real numbers p and q such that

$$p \log h_\rho(m) + q \log h_\rho(l) = 2\pi i.$$

Deform $h_{\rho_0}(l)$ a little bit and it becomes a non-identity hyperbolic element and so $h_{\rho_0}(m)$ must also be a non-identity hyperbolic element because it commutes.

11.2 Ideal Simplex Parameters

Let M satisfy the same conditions as in the previous section. Suppose that K is a simplicial complex with 1-vertex such that $M \cong K \setminus \{\text{vertex}\}$. We will think of all the simplices of K as ideal hyperbolic simplices and let them have simplex parameters $z_1, \dots, z_n$ such that $z_i \in \mathbb{C} \setminus \{0, 1\}$ and $\mathrm{Im}(z_i) > 0$.

Question 5. *What are the conditions on the simplex parameters $z_i, \dots, z_n$ so that K has a complete hyperbolic structure?*

12 Lecture 12: Geometrization Conjecture

Let M be closed, orientable and write $M = M_1 \# M_2 \# \dots \# M_r$ for $r \geq 1$ and M_i irreducible.

Theorem 12.1 (Geometrization Conjecture)**.** *Any irreducible closed orientable 3-manifold M is diffeomorphic to $M = \bigcup N_i$ where the N_i are either*

(i) a S^1-bundle over a 2-dimensional orbifold;

(ii) or a hyperbolic manifold.

12.1 Reviewing surface bundles over S^1

Much of this is repeat information from Section 1.1.

Definition 12.2. M closed, orientable 3-manifold which fibers over S^1 given by $\Sigma_g \to M \to S^1$. Then $M \cong M_f$ for some homeomorphism f of Σ_g. This is determined by the monodromy of the foliation and it determines a representation

$$\rho : \pi_1 S^1 \to \pi_0(\mathrm{Diff}^+ \Sigma_g) =: \mathrm{MCG}(\Sigma_g), \qquad \omega \mapsto [f].$$

Remark 12.3. –

1. If $f \sim h$ are homotopic, then $M_f \cong M_h$ are homeomorphic. Given this, we can refer to $M_{[f]}$ for a homotopy class of homeos.

2. If $[f]$ is conjugate to $[h]$ in $\mathrm{MCG}(\Sigma_g)$, then $M_f \cong M_h$.

3. Number 2 is NOT an iff. There are two manifolds M_f and M_h where f and h are not conjugate in $\mathrm{MCG}(\Sigma_g)$, but $M_f \cong M_h$. However, this will *not* be a fiber-preserving map.

"*every 3-manifold is a surface bundle*"– Probably Benson

For an algebraic variety X, there is a closed submanifold D such that

$$\begin{array}{ccc} \Sigma_{g,n} & \longrightarrow & X \setminus D \\ & & \downarrow \\ S^1 & \dashrightarrow & Y \end{array}$$

Actually I don't know what this statement is trying to say.

12.2 Surface bundles with fiber S^2 and T^2

We are interested in the diffeomorphism types of $\Sigma \to M \to S^1$ depending on the genus of Σ. We start with the two smallest cases.

- **g = 0:** If $\Sigma = S^2$, then there are two options for M up to diffeomorphism. Either $M \cong S^2 \times S^1$ or $M \cong S^2 \widetilde{\times} S^1$. The second case is non-orientable (i.e. the gluing map is orientation reversing).

- **g = 1:** Recall that $\mathrm{MCG}(T^2) \cong \mathrm{SL}_2\,\mathbb{Z}$ in a canonical way once we choose a basis for the first homology $H_1(T^2, \mathbb{Z})$. For $A \in \mathrm{SL}_2\,\mathbb{Z}$, we can build the mapping torus M_A. Every torus bundle over the circle is in this form. The geometry of the bundle is determined by the dynamics of the map A.

Proposition 12.4. *For each $A \in \mathrm{SL}_2\,\mathbb{Z}$ exactly one of the following is true:*

1. *A is of finite order;*

2. *A is conjugate to $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} -1 & n \\ 0 & -1 \end{pmatrix}$ for $n \in \mathbb{Z}$;*

3. $|\operatorname{tr} A| > 2$. *In this case, A either has two rational eigenvalues or two irrational eigenvalues. We call matrix of this form* Anosov.

Exercise: Prove this result. Start with the characteristic polynomial $X^2 - (\operatorname{tr} A)X + 1 = 0$.

Theorem 12.5. *Let $T^2 \to N \to S^1$ be a(n orientable) torus bundle over the circle. Then $N \cong M_A$ (diffeomorphic) for some $A \in \mathrm{SL}_2\mathbb{Z}$. There are three cases:*

1. *If A is finite order then M_A has a finite cover by $\widetilde{M_A} \cong T^2 \times S^1 \cong T^3$. Geometrically, $T^3 = \mathbb{E}^3/\mathbb{Z}^3$ has Euclidean geometry so M_A does as well.*

2. *If A is conjugate to $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$, let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then*

$$\pi_1 M_A \cong \mathbb{Z}^2 \rtimes_A \mathbb{Z} = \{t, x, y \mid [x, y] = 1, txt^{-1} = x^a y^c, tyt^{-1} = x^b y^d\}.$$

   In this case, $M_A = Nil/\Gamma$ for a discrete group Γ.

3. *A is Anosov and it preserves a pair of transverse measured foliations on the torus. In this case, $M_A \cong (\mathbb{R}^2 \rtimes \mathbb{R})/(\mathbb{Z}^2 \rtimes_A \mathbb{Z})$ and it has Solv geometry "sol".*

Note: With option (2), we get an essential torus. This follows because there is an invariant essential simple closed curve in T^2 under the action by A. After this curve is suspended, it becomes an essential torus in M_A. It helps to work this out with the simple map $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$.

Remark 12.6 (More on Anosov maps). Suppose that A acting on $\mathbb{R}^2$ is an Anosov map. This happens if and only if $|\operatorname{tr} A| > 2$ and it follows that A has two real eigenvalues. Then A descends to φ_A acting on $\mathbb{R}^2/\mathbb{Z}^2 \cong T^2$. Since A is Anosov, it leaves invariant two transverse foliations u and s of $\mathbb{R}^2$; these are exactly given by lines parallel to the eigenvector directions. Let s be the *stable foliation* with eigenvalue $\lambda < 1$ and u be the *unstable foliation* with eigenvalue $\lambda^{-1} > 1$. Note that the eigenvalues are inverses of each other since $\det(A) = 1$. These foliations also descend to a pair of transverse foliations on T^2 given by $\mathcal{F}^u$ and $\mathcal{F}^s$.

For each foliation, we define a transverse measure $\mu_{u/s} : \{\text{arcs transverse to foliation } \mathcal{F}^{u/s}\} \to \mathbb{R}$ which is inherited from the Euclidean metric on the torus. Let $\gamma(t)$ be a parameterized arc and $e_{u/s}$ be the stable/unstable eigenvector. Then we define

$$\mu_{u/s}(\gamma) = \int \langle \gamma'(t), e_{u/s} \rangle_{T^2}^{\frac{1}{2}} \, dt.$$

By construction of this transverse measure, we see that φ_A acts nicely on it.

$$\varphi_A(\mathcal{F}^u, \mu^u) = (\mathcal{F}^u, \lambda^{-1}\mu^u) \quad \text{and} \quad \varphi_A(\mathcal{F}^s, \mu^s) = (\mathcal{F}^s, \lambda\mu^s)$$

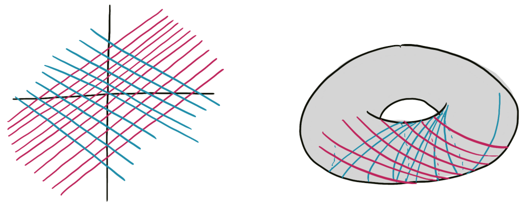

Figure 19: An Anosov map on the torus has two invariant transverse measured foliations.

We call the foliations with these transverse measures the *set of invariant transverse measured foliations*. The existence of such foliations are often used to characterize Anosov maps.

Remark 12.7. *Nil* is the geometry of upper triangular matrices with 1's on the diagonal. This is called the Heisenburg group. $N(\mathbb{R})/N(\mathbb{Z})$.

12.3 Nielsen-Thurston Classification of homeomorphisms for higher genus surfaces.

"The best source for this is Primer." – Benson

Now that we've seen the story for tori, let's take a look at higher genus surfaces. We start by classifying all the homeomorphisms on Σ_g in an analogous way to our torus result.

Theorem 12.8 (Nielsen-Thurston Classification)**.** *For all* $f \in \mathrm{Homeo}^+(\Sigma_g)$*, there is* $h \in \mathrm{Homeo}^+(\Sigma_g)$ *with* $h \sim f$ *and* h *satisfies exactly one of the following:*

1. *(finite order).* $h^d = \mathrm{Id}$ *for some* $d \in \mathbb{Z}_{\geq 1}$.[15]
2. *(reducible). There is a finite set of (isotopy classes of) essential simple closed curves* $\gamma_1, \gamma_2, \ldots, \gamma_k$ *essential such that* $h(\{\gamma_i\}) = \{\gamma_i\}$.
3. *(pseudo-Anosov) There exists a pair of transverse singular measured foliations* $\mathcal{F}^s$ *(stable) and* $\mathcal{F}^u$ *(unstable) and a* $\lambda > 1$ *such that*

$$h(\mathcal{F}^s, \mu^s) = (\mathcal{F}^s, \lambda^{-1}\mu^s), \qquad h(\mathcal{F}^u, \mu^u) = (\mathcal{F}^u, \lambda\mu^u)$$

[15]Note that there is some subtly here. The Nielsen Realization Theorem asserts that if $f^d \sim \mathrm{Id}$, then $\exists h \sim f$ such that $h^d = \mathrm{Id}$. It is nontrivial to prove this

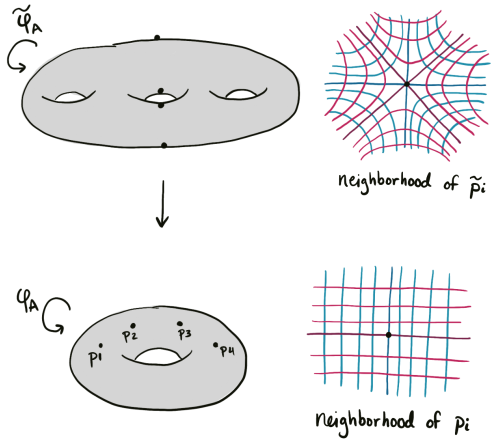

Figure 20: One way to build a pseudo-Anosov map on a higher genus surface is to lift one from an Anosov map. The points $p_1, \ldots, p_4$ are fixed under the Anosov map on the torus and we can use a branched covering $\Sigma_3 \to T^2$ with $p_1, \ldots, p_4$ as critical values.

Remark 12.9 (on singular foliations)*.* By Poincaré-Hopf, there is no non-vanishing line field on Σ_g for $g \geq 2$ so we cannot foliate by 1-manifolds like in the torus case. Hence, for $g \geq 2$, some of the leaves in the foliations are trees with branch points.

Example 12.10*.* Take $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{10}$ to be the tenth power of the cat map and $\varphi_A \in \mathrm{Diff}(T^2)$ like in the discussion of Anosov maps. First, notice that all rational points on T^2 are periodic under this map.

Next, pick fixed points $\{p_1, \ldots, p_r\}$ of A (these points will be periodic under cat map with period 10). Then form a branched cover $\Sigma_g \to T^2$ branched over $\{p_1, \ldots, p_r\}$. Then we can lift A to a homeomorphism of Σ_g under this branched cover. See Figure 20.

Exercise: Prove that all rational points on T^2 are periodic under A.

For any Riemannian metric s on Σ_g, there is some pseudo-Anosov map h such that for all simple closed curves $\gamma \subset \Sigma_g$, we have $\lim_{n\to\infty} \sqrt[n]{\ell_s(h^n(\gamma))} = \lambda(h)$. I am not sure what the bigger context of this fact is/why it was mentioned.

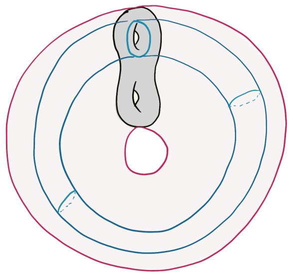

Figure 21: If φ is reducible, then a power of it fixes an essential closed curve. The suspension of this curve in the mapping tours is an essential torus.

13 Lecture 13: Surface Bundles in higher genus

Last time we classified the different homeomorphisms on a genus g surface Σ_g, now we consider what the mapping torus looks like for each class of homeomorphism.

Theorem 13.1 (Thurston early 80s). *Let $\varphi \in \mathrm{MCG}(\Sigma_g)$ for $g \geq 2$. Let M_φ be the mapping torus of φ. Then the geometry of M_φ depends on the classification of φ according to Theorem 12.8.*

1. *If φ is finite order then $\Sigma_g \times S^1$ finitely covers M_φ and M_φ has geometry $\mathbb{H}^2 \times \mathbb{E}^1$.*
2. *If φ is reducible then M_φ contains a π_1-injective torus $T^2 \hookrightarrow M_\varphi$ (see Figure 21). In this case, we can cut along this torus and reapply a version of this theorem for surfaces with boundary.*
3. *φ is pseudo-Anosov iff $M_\varphi \cong_{diff} \mathbb{H}^3/\Gamma$ for Γ a discrete co-compact subgroup of $\mathrm{PSL}(2, \mathbb{C})$. In this case $\Gamma \cong \pi_1 M_\varphi$ and M_φ is a hyperbolic manifold.*

proof of $\Rightarrow$ direction for (3). If φ is not pseudo-Anosov, then it is either reducible or finite-order. In either case, $\pi_1 M_\varphi$ contains a $\mathbb{Z} \times \mathbb{Z}$ subgroup and M_φ cannot have a hyperbolic structure by the Geometrization Theorem 3.1.

Exercise: Why does finite order and reducible imply that there is a $\mathbb{Z}\times\mathbb{Z}$ subgroup?

□

Goal 13.2. *Walk through Thurston's proof of the other direction $\Leftarrow$ for (3).*

We will prove this direction in Section 15.2.

Remark 13.3. Remarks on the previous theorem.

- The $\mathbb{H}^3$-structure on M_φ for φ pseudo-Anosov is locally homogeneous but it doesn't "look" like it. The fiber bundle structure gives it a natural foliation with the transverse direction to the fibers somewhat distinguished.

- There is an equivalent rep theory description for having a hyperbolic structure on M_φ. Note that $\pi_1 M$ is a semidirect product given by $\pi_1 M_\varphi = \pi_1 \Sigma_g \rtimes_{\varphi_*} \mathbb{Z}$.

  So M_φ has a hyperbolic structure if and only if there is a (discrete, faithful, cocompact) representation

  $$\rho : \pi_1 \Sigma_g \rtimes_{\varphi_*} \mathbb{Z} \to \mathrm{PSL}(2, \mathbb{C}) \cong \mathrm{Isom}^+(\mathbb{H}^3).$$

  For such a representation, M_φ is homotopy equivalent to $\mathbb{H}^3 / \operatorname{Im}(\rho)$ and M_φ is hyperbolic with hyperbolic structure $\mathbb{H}^3 / \operatorname{Im}(\rho)$.

  We can go one step further. Note that $\mathrm{MCG}(\Sigma_g)$ acts on the set

  $$\{\psi : \pi_1 \Sigma_g \to \mathrm{PSL}(2, \mathbb{C}) \text{ discrete, faithful}\}/\text{conjugacy} \subset \mathrm{Hom}(\pi_1 \Sigma_g, \mathrm{PSL}(2, \mathbb{C})).$$

  A representation ρ as desired exists if and only if there is some (discrete, faithful) representation of $\pi_1 \Sigma_g$ fixed by the action of φ on this set. This can be rephrased in the following way.

  Question 6. *Given a representation ψ, when can we extend it to a representation ρ of the whole group?*

  We have then reduced the problem of finding a hyperbolic structure to a fixed point problem! This motivates us to study surface subgroups of $\mathrm{PSL}(2, \mathbb{C})$.

13.1 Kleinian Groups

Definition 13.4. A *Kleinian group* is a discrete subgroup $\Gamma < \mathrm{Isom}^+(\mathbb{H}^3) = \mathrm{M\ddot{o}b}(\hat{\mathbb{C}}) = \mathrm{PSL}_2(\mathbb{C})$.

There are many reasons to study such groups.

1. Studying discrete subgroups of the group of Möbius transformations $\mathrm{M\ddot{o}b}(\hat{\mathbb{C}})$ emphasizes the **Complex Analytical/Dynamic aspect**.

2. Studying discrete subgroups of $\mathrm{PSL}_2(\mathbb{C})$ emphasizes the **Lie group/matrix group theoretic aspect**.

3. Studying discrete subgroups of $\mathrm{Isom}(\mathbb{H}^3)$ emphasizes the **Hyperbolic Geometry aspect**.

We will be focusing on the third perspective and hence, consider only torsion-free Kleinian groups as they give rise to hyperbolic 3-manifolds - $M = \mathbb{H}^3/\Gamma$.

Further, for simplicity and ease of computation, we consider finitely generated groups.

Example 13.5 (Fuchsian Groups). Suppose we have a discrete faithful representation $\rho : \pi_1\Sigma_g \to \mathrm{PSL}(2,\mathbb{R}) \hookrightarrow \mathrm{PSL}(2,\mathbb{C})$. Then $\rho : \pi_1\Sigma_g \to \mathrm{PSL}(2,\mathbb{R})$ gives a hyperbolic structure on Σ_g, i.e., $\mathbb{H}^2/\rho(\pi_1\Sigma_g)$. With the representation into the whole group $\mathrm{PSL}(2,\mathbb{C})$, we can take the quotient and get a geometry on $\Sigma_g \times \mathbb{R}$.

$$\Sigma_g \times \mathbb{R} \cong \mathbb{H}^3/\rho(\pi_1\Sigma_g)$$

The action by $\rho(\pi_1\Sigma_g)$ on $\mathbb{H}^3$ fixes a geodesic hyperplane which is one copy of $\widetilde{\Sigma_g}$. Then $\mathbb{H}^3$ is foliated by parallel copies of this hyperplane in the same way that $\Sigma_g \times \mathbb{R}$ is foliated by $\Sigma_g \times \{t\}$.[16] See Figure 22.

Now suppose that we want to put a hyperbolic structure on M_f where M_f is a 3-manifold fibering over a circle, with fiber Σ_g. Note that $\pi_1 M = \pi_1(\Sigma_g) \rtimes_{f_*} \mathbb{Z} = \langle \pi_1\Sigma_g, t | tat^{-1} = f_*(a)\ \forall a \in \pi_1(\Sigma_g)\rangle$.

This is equivalent in the groups world to extending ρ to a discrete faithful representation of $\pi_1\Sigma_g \rtimes \mathbb{Z}$. Suppose that we have $\overline{\rho} : \pi_1\Sigma_g \rtimes \mathbb{Z} \to \mathrm{PSL}(2,\mathbb{C})$ and that $\mathbb{Z}$ is generated by t. Then $\mathbb{H}^3$ is still foliated by planes invariant under the action by $\overline{\rho}(\pi_1\Sigma_g)$. The action of t on $\mathbb{H}^3$ must be a hyperbolic translation along a geodesic transverse to this foliation. And because of the semi-direct product structure, we have $tat^{-1} \in \pi_1\Sigma_g$ for all $a \in \pi_1\Sigma_g$. But geometrically, this doesn't work if t acts by hyperbolic translation. In this case then, $\overline{\rho}(\pi_1\Sigma_g)$ *cannot* leave invariant a *totally geodesic* hyperplane.

Exercise: Prove that we cannot have $\overline{\rho}(tat^{-1}) \in \overline{\rho}(\pi_1\Sigma_g)$ for all $a \in \pi_1\Sigma_g$ and t a hyperbolic translation.

Definition 13.6. The *limit set* of a Kleinian group $\Gamma < \mathrm{PSL}(2,\mathbb{C})$ is $\Lambda_\Gamma := \overline{\Gamma \cdot x} \cap \partial\mathbb{H}^3$ where you can pick any $x \in \mathbb{H}^3$.

Exercise: A different choice of x gives the same set.

[16] We will talk more about foliations in Section 17.

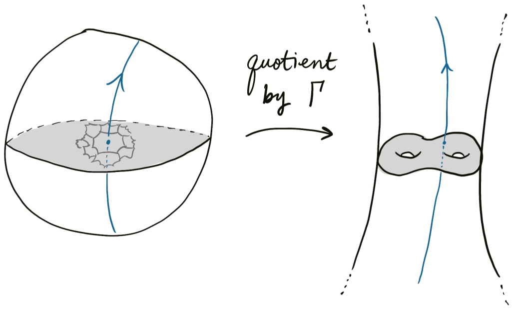

Figure 22: A Fuchsian group leaves invariant a totally geodesic hyperplane.

Figure 23: A figure of a dihedral tetrahedron and the Sierpinkski carpet

Remark 13.7 (Action on the boundary sphere $\partial\mathbb{H}^3$)*.* Remember that we define

$$\partial\mathbb{H}^3 = \{\text{equivalence classes of geodesics}\}.$$

And then $\overline{\mathbb{H}^3} := \mathbb{H}^3 \cup \partial\mathbb{H}^3$. For Kleinian groups, we look for Γ which act properly discontinuously on $\mathbb{H}^3$ but they (in general) will NOT do so on $\partial\mathbb{H}^3$. On this set, Γ will act by conformal automorphisms $\mathrm{Conf}(\partial\mathbb{H}^3)$. $\partial\mathbb{H}^3$ can be identified with $\hat{\mathbb{C}}$, thus making the action on the boundary to be actions by Möbius maps on the Riemann sphere.

Example 13.8. For a standing assumption, we suppose that Γ is not elementary (i.e. that $|\Lambda_\Gamma| > 2$).

1. *Fuchsian groups:* $\Gamma < \mathrm{PSL}(2,\mathbb{R}) < \mathrm{PSL}(2,\mathbb{C})$. In this case, Λ_Γ is a round circle.

2. $\Gamma < \mathrm{PSL}(2,\mathbb{C})$ is a lattice (discrete, free, cocompact). Then $\Lambda_\Gamma = S^2_\infty$.

   Proof: Tile $\mathbb{H}^3$ with fundamental domains. Let x be an arbitrary point in the interior $\mathbb{H}^3$. Choose $y \in \partial\mathbb{H}^3$ and connect x to y with a geodesic. Apply group elements for each tile that the geodesic passes through. This gives a sequence of points in the orbit of x that converge to y.

3. Let $\Gamma = \langle R_1, R_2, R_3, R_4\rangle$ be reflections across the four faces of a tetrahedron with dihedral angles $\frac{2\pi}{n}$. Then the limit set is the Sierpinski carpet (see Figure 23).

Note that Λ_Γ is a closed and Γ-invariant subset of S^2_∞. This means that Λ_Γ is minimal set under the action by Γ on S^2_∞ (i.e. the orbit of every point is dense in Λ_Γ).

Proposition 13.9. *Λ_Γ is the unique minimal, nonempty, closed, Γ-invariant subset of S^2.*

Corollary 13.10. *If $H < \Gamma$ is a normal subgroup, then $\Lambda_H = \Lambda_\Gamma$.*

> **Exercise:** Prove this. Λ_H is Γ-invariant implying that $\Lambda_\Gamma \subset \Lambda_H$ but also if $\Gamma_1 < \Gamma_2$, then $\Lambda_{\Gamma_1} \subset \Lambda_{\Gamma_2}$. (Where are we using normal here?)

Corollary 13.11 (Cannon-Thurston Map)**.** *Suppose that $\Gamma \cong H \rtimes \mathbb{Z}$ and $H \cong \pi_1\Sigma_g$ and Γ is a cocompact Kleinian group. This is the case of $\Gamma \cong \pi_1 M_\varphi$ for φ pseudo-Anosov. Then we know $\Lambda_H = \Lambda_G = S^2_\infty$ by Proposition 13.9. But also, the limit set of $H = \pi_1\Sigma_g$ is a circle, the boundary of $\mathbb{H}^2$. So Λ_G is a circle whose closure is the whole sphere— a* space-filling curve. *The map $S^1 \to \Lambda_G$ is called the* Cannon-Thurston Map.

Geometrically, there is a stack of disks foliating $\mathbb{H}^3$. Each of them is $\widetilde{\Sigma_g} \times \{t\}$ and it has a hyperbolic structure. The circle at the boundary for each disk is the space-filling circle of the Cannon-Thurston map.

14 Lecture 14: Limit Sets and Bowen Rigidity

As we discussed in the last lecture, the limit set Λ_Γ of a Kleinian group is a minimal, closed, Γ-invariant subset of S^2_∞. We now consider the action of Γ on the other part of S^2_∞.

Proposition 14.1. *Let $C \subset \partial\mathbb{H}^3$ be a closed Γ-invariant set with $|C| > 2$. Then, the Γ-action on $\partial\mathbb{H}^3 \setminus C$ is properly discotinuous.*

Proof. Let $\mathcal{C} = \text{ConvexCore}(C)$. The nearest point retraction $r\colon \overline{\mathbb{H}^3} \to \mathcal{C}$ is the map that sends x to the point $r(x) \in \mathcal{C}$ that is closest to x. If $x \in \partial\mathbb{H}^3$, we interpret $r(x)$ as the first point of $\mathcal{C}$ that is contained in some horosphere centred at x. Using the geodesic from x to $r(x)$ we can construct a natural deformation retraction of $\overline{\mathbb{H}^3}$ onto the closed convex set $\mathcal{C}$. Moreover, r is Γ-invariant.

r sends $\mathbb{H}^3 \cup (\partial\mathbb{H}^3 \setminus C)$ to $\mathcal{C} \setminus C$ and commutes with Γ. The action of Γ on $\mathcal{C} \setminus C$ is properly discontinuous (since it is contained in $\mathbb{H}^3$), so the action on $\mathbb{H}^3 \cup (\partial\mathbb{H}^3 \setminus C)$ is also properly discontinuous. □

We call the set $\Omega_\Gamma := S^2_\infty \setminus \Lambda_\Gamma$ the *domain of discontinuity.* Then the manifold (with boundary if $\Omega_\Gamma \neq \emptyset$) is given by $M_\Gamma := (\mathbb{H}^3 \cup \Omega_\Gamma)/\Gamma$. The boundary of this manifold is given by $\partial M_\Gamma = \Omega_\Gamma/\Gamma$.

Example 14.2. We have already seen at least one instance of the domain of discontinuity.

1. Consider a Fuchsian group $\rho : \pi_1\Sigma \to \mathrm{PSL}(2,\mathbb{R}) \hookrightarrow \mathrm{PSL}(2,\mathbb{C})$. In this case, Λ_Γ is a round circle and $\Omega_\Gamma = \Omega^+ \cup \Omega^-$ has two connected components. Then $\mathbb{H}^3/\Gamma \cong \Sigma_g \times \mathbb{R}$ and its compactification is $\Sigma_g \times [0,1]$. The two boundary components $\Sigma_g \times 0$ and $\Sigma_g \times 1$ are quotients Ω^+ and Ω^- by the action of Γ.

2. We also have *quasi-Fuchsian* groups. Suppose that $\Gamma < \mathrm{PSL}(2,\mathbb{C})$ is conjugate to a Fuchsian group Γ' by a quasi-conformal map f (i.e. $\Gamma = f \cdot \Gamma' \cdot f^{-1}$). This happens if and only if Λ_Γ is a quasi-circle Jordan curve $S^1 \hookrightarrow S^2$. One famous example of the limit set of quasi-Fuchsian group is affectionately called "Mickey Mouse".

Example 14.3 (Mickey Mouse). Put a hyperbolic structure on Σ_g and suppose that we have a representation of a Fuchsian group $\rho_0 : \pi_1\Sigma_g \to \mathrm{PSL}(2,\mathbb{R}) < \mathrm{PSL}(2,\mathbb{C})$ and let $\Gamma = \rho_0(\pi_1\Sigma_g)$. The limit set Λ_Γ is a round circle bounding a copy of $\mathbb{H}^2$. The lift of a separating curve γ on Σ_g is a lamination of this $\mathbb{H}^2$ (see Figure 24). We want to "bend" the the $\mathbb{H}^2$ along all the lifts of γ. Let γ act with matrix

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

Since γ is separating, represent Σ_g as the union of two subsurfaces S_1 and S_2 glued along boundary γ. We can decompose $\pi_1\Sigma_g$ according to these subsurfaces.

$$\rho_0 : \pi_1\Sigma_g = \pi_1 S_1 *_\gamma \pi_1 S_2 \to \mathrm{PSL}(2,\mathbb{R})$$

Then the bending of angle t follows from choosing a matrix $A_t \in C_{\mathrm{PSL}(2,\mathbb{C})}(A)$ which commutes with the hyperbolic translation $A = \rho_0(\gamma)$ and this can either be elliptic or hyperbolic. We have a 2-parameter family of deformations given by the elliptic and hyperbolic components.

$$A_t = \begin{pmatrix} \lambda^s & 0 \\ 0 & \lambda^{-s} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Then we define a "bent" representation

$$\rho_t : \alpha \mapsto A_t \cdot \rho(\alpha) \cdot A_t^{-1}.$$

Note that the action of γ is fixed because A_t commutes with A but the actions by other curves are perturbed. The resulting group $\rho_t(\pi_1\Sigma_g)$ is quasi-Fuchsian and the quotient manifold is quasi-Fuchisian.

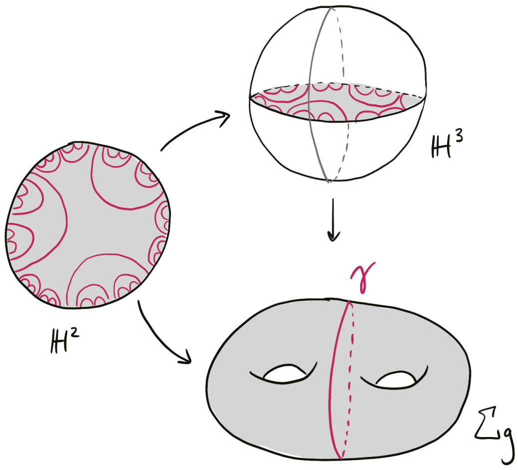

Figure 24: The lift of a separating curve γ in the surface Σ_g is a lamination in $\mathbb{H}^2$.

Theorem 14.4 (Bowen Rigidity Theorem)**.** *If* Γ *is quasi-Fuchsian, then the* Hausdorff dimension *of the limit set satisfies* $1 \leq \mathrm{HD}(\Lambda_\Gamma) < 2$ *with equality to* 1 *if and only if* Γ *is Fuchsian.*

A generalized narrative for math: Look at $\mathrm{Hom}(G_1, G_2)$

1. G_1 finite, $G_2 = \mathrm{GL}_n\,\mathbb{C}$ is representations of finite groups
2. $G_1 = \pi_1 M$, $G_2 = \mathrm{Isom}(\mathbb{H}^n)$ gives hyperbolic metrics on M.
3. $G_1 = \pi_1 M$ and $G_2 =$anything, then we get G_2-covers of M.
4. If $G_1 =$Lie group and $G_2 = \mathrm{GL}_n\,\mathbb{C}$, then we have rep theory of Lie groups.
5. $G_1 = Gal(\overline{\mathbb{Q}}|\mathbb{Q})$, $G_2 = \mathrm{GL}_2\,\mathbb{Q}_p$ is number theory.

15 Lecture 15: Deformation Spaces

15.1 Teichmuller spaces and Moduli spaces of surfaces

First with a torus: We define the following *Moduli space* of marked tori.

$$\mathcal{M}_1 = \{\text{unit area flat tori}\}/\text{isometry} = \{\text{unit co-area lattices } \Lambda < \mathbb{R}^2\}/\text{isometry}$$

We can also consider the universal cover of $\mathcal{M}_1$ called the Teichmüller space.[17]

$$\mathcal{T}(\Sigma_{1,1}) = \{\text{unit area ordered bases } B = \{v_1, v_2\}\}/\operatorname{SO}_2(\mathbb{R})$$

In the Teichmüller space, we care about the generators for the lattice (a *marked* flat structure). But in the moduli space, we don't keep track of the generators. So there is a natural map from Teichmüller space to moduli space where we forget the generators.

$$\mathcal{T}(\Sigma_{1,1}) \to \mathcal{M}_1, \qquad \{v_1, v_2\} \mapsto \mathbb{Z}\langle v_1, v_2\rangle$$

Notice that $\operatorname{SL}_2\mathbb{Z}$ acts on $\mathcal{T}(\Sigma_{1,1})$ by matrix multiplication. For $A \in \operatorname{SL}_2\mathbb{Z}$, we define an action on the basis $(v_1, v_2) \mapsto (Av_1, Av_2)$. This action is transitive on all possible basis sets for a given lattice $\mathbb{Z}\langle v_1, v_2\rangle$. Hence, we have that $\mathcal{T}(\Sigma_{1,1})/\operatorname{SL}_2\mathbb{Z} = \mathcal{M}_1$.

Example 15.1. Consider the following basis sets.

$$B_1 = \{(1,0),(0,1)\}, \quad \text{and} \quad B_2 = \{(1,0),(1,1)\}$$

These ordered basis represent different points in $\mathcal{T}(\Sigma_{1,1})$ but the same point in the moduli space because they generate the same lattice in $\mathbb{R}^2$.

Proposition 15.2. *$\mathcal{T}(\Sigma_{1,1}) = \mathbb{H}^2$.*

To see this, start with any ordered basis v_1, v_2. Points in $\mathcal{T}(\Sigma_{1,1})$ are normalized so the area of the parallelogram spanned by v_1 and v_2 has unit area. Instead, we switch to normalizing to make $|v_1| = 1$; this will give the same space. Now rotate v_1 to be aligned with the positive real axis so it matches up with the vector $(1, 0)$. Then there is a one parameter family of choices for v_2. Note that v_2 must lie in the upper half plane because $\det[v_1 v_2]$ is positive. The position of v_2 parametrizes $\mathcal{T}(\Sigma_{1,1})$ so as sets, it is equivalent to the upper half plane. The different choices of v_2 also give a topology on $\mathcal{T}(\Sigma_{1,1})$ to make it equivalent to $\mathbb{H}^2$ as topological spaces as well.

Exercise: Finish the proof and prove that under this identification, $\operatorname{SL}_2\mathbb{Z}$ acts on $\mathcal{T}(\Sigma_{1,1})$ in the same way that $\operatorname{SL}_2\mathbb{Z}$ acts on $\mathbb{H}^2$ by Möbius Transformations. Then see that $\mathcal{T}(\Sigma_{1,1})$ is the upper half plane tessellated by ideal $(2, 3, \infty)$ triangles and the moduli space $\mathcal{M}_1$ is the quotient by this symmetry and we get a spherical orbifold with cone points of order $2, 3, \infty$. See Figure 25.

[17]It is worth noting that Oswald Teichmüller was a supporter of the Nazi party. We continue to use his name out of convention.

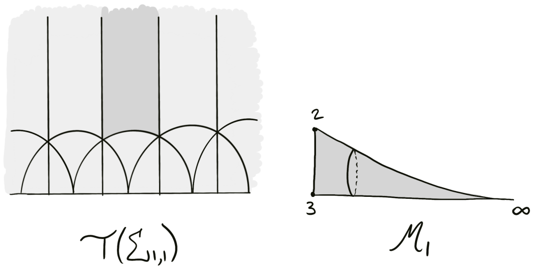

Figure 25: (left) The Teichmüller space tessellated by $(2, 3, \infty)$ triangles and (right) the Moduli space which is a quotient of $\mathcal{T}(\Sigma_{1,1})$ by the tessellation.

We give another definition of the Teichmüller space of the torus $\mathcal{T}(T^2)$ which will generalize to higher genus surfaces.

$$\mathcal{T}(T^2) = \big\{(X, f) : \ X \text{ is a flat torus with unit area, } f : T^2 \to X\big\} / \sim$$

The equivalence relation $\sim$ is defined by $(X_1, f_1) \sim (X_2, f_2)$ if there is an isometry $\phi : X_1 \to X_2$ such that the following diagram commutes up to homotopy.

$$\begin{array}{ccc} T^2 & \xrightarrow{f_1} & X_1 \\ & \searrow^{f_2} & \downarrow \phi \\ & & X_2 \end{array}$$

Higher genus $g \geq 2$**:** The moduli space for Σ_g can be defined in a few equivalent ways.

$$\begin{aligned} \mathcal{M}_g &= \{\text{complex structures on } \Sigma_g\}/\text{biholomorphisms} && (1) \\ &= \{\text{Riemannian metrics on } \Sigma_g\}/\text{conformal maps} && (2) \\ &= \{\text{constant } -1 \text{ curvature metrics on } \Sigma_g\}/\operatorname{Isom} && (3) \\ &= \{\text{smooth genus } g \text{ curve}\}/\text{birational equivalence} && (4) \end{aligned}$$

The equivalence from line (1) to (2) is given by *isothermal coordinates*. The equivalence from (1) to (4) follows from the Kodaira Embedding Theorem. For the equivalence of (1) and (3), we know that the universal cover of a surface Σ_g with a complex structure has a universal cover homeomorphic to $\mathbb{R}^2$. Then we can apply the Uniformization Theorem and the fact that $\operatorname{Aut}(\mathbb{D}) \cong \operatorname{PSL}(2, \mathbb{R}) \cong \operatorname{Isom}^+(\mathbb{H}^2)$ to recover a hyperbolic metric in the conformal class.

Definition 15.3 (Teichmüller space for Σ_g)**.** Let $g \geq 2$. We define the *Teichmüller space* of Σ_g as the space of *marked complex structures* (X, f) up to an equivalence relation.

$$\mathcal{T}(\Sigma_g) = \{(X, f) : X \text{ hyperbolic surface homeomorphic to } \Sigma_g, f : \Sigma_g \to X \text{ a diffeomorphism}\} / \sim$$

We say that $(X_1, f_1) \sim (X_2, f_2)$ if there is an isometry ϕ such that the following diagram commutes up to homotopy.

$$\begin{array}{ccc} \Sigma_g & \xrightarrow{f_1} & X_1 \\ & \overset{f_2}{\searrow} & \downarrow \phi \\ & & X_2 \end{array}$$

We can also define $\mathcal{T}(\Sigma_g)$ in a more algebraic way using discrete, faithful representations of the fundamental group.

$$\mathcal{T}(\Sigma_g) \cong DF(\pi_1\Sigma_g, \mathrm{PSL}(2, \mathbb{R})) \Big/ \mathrm{PSL}(2, \mathbb{R})$$

Then the outer automorphisms $\mathrm{Out}(\pi_1\Sigma_g)$ acts on representations on the left and $\mathrm{Out}(\pi_1\Sigma_g) \cong \mathrm{MCG}(\Sigma_g)$. The moduli space $\mathcal{M}_g$ is the quotient of $\mathcal{T}(\Sigma_g)$ by this action.

Proposition 15.4. $\mathcal{M}_g = \mathcal{T}(\Sigma_g) / \mathrm{MCG}(\Sigma_g)$

15.2 Proof of hyperbolization for $\Sigma_g \to M^3 \to S^1$

We finally return to Goal 13.2 to prove the second direction for Thurston's theorem on hyperbolic mapping tori.

Theorem 15.5 (Thurston)**.** *Let $g \geq 2$. Let $\psi \in \mathrm{MCG}(\Sigma_g)$ be pseudo-Anosov. Then the mapping torus M_ψ is $\mathbb{H}^3/\Gamma$ for some $\Gamma \cong \pi_1(M_\psi)$.*

Remark 15.6. There is a similar theorem for $\psi \in \mathrm{MCG}(\Sigma_{g,n})$ where $\Sigma_{g,n}$ has genus g and n boundary components. Then M_ψ has a complete finite volume hyperbolic metric with torus boundary components if ψ is pseudo-Anosov.

Here is the proof idea. We want to build a discrete faithful representation $\rho : \pi_1 M_\psi \to \mathrm{PSL}(2, \mathbb{C})$ and this will give as a hyperbolic structure. First, we examine this fundamental group a little more closely.

$$\pi_1 M_\psi = \pi_1\Sigma_g \rtimes_\psi \mathbb{Z}$$

Where if $\mathbb{Z} = \langle t \rangle$ and $\alpha \in \pi_1\Sigma_g$, then $t\alpha t^{-1} = \psi(\alpha)$.

Exercise: Work out why this is true. It follows by lifting to the cover $\Sigma_g \times \mathbb{R}$ and then noticing that the deck group is given by iterates of ψ.

Given a representation $\rho : \pi_1 M_\psi \to \mathrm{PSL}(2, \mathbb{C})$, we get an inherited representation $\tilde{\rho} : \pi_1\Sigma_g \to \mathrm{PSL}(2, \mathbb{C})$.

$$\pi_1\Sigma_g \xrightarrow{i} \pi_1 M_\psi \xrightarrow{\rho} \mathrm{PSL}_2\mathbb{C}, \qquad \tilde{\rho} = \rho \circ i$$

We apply this to the earlier observation to get the following equivalence for $\alpha \in \pi_1\Sigma_g$.

$$\rho(\psi_*(\alpha)) = \rho(t\alpha t^{-1}) = \rho(t)\rho(\alpha)\rho(t)^{-1}$$

Instead of looking for ρ straightaway, we can look for $\tilde{\rho}$ such that $\tilde{\rho}$ and $\tilde{\rho} \circ \psi_*$ are conjugate representations $\pi_1\Sigma_g \to \mathrm{PSL}(2, \mathbb{C})$. In particular, we want to find such a representation which is discrete and faithful. In order to think about representations $\pi_1\Sigma_g \to \mathrm{PSL}(2, \mathbb{C})$, we define the following space.[18] One should picture the manifolds N as homeomorphic to $\Sigma_g \times \mathbb{R}$.

$$\mathrm{AH}(\Sigma_g) := \left\{ (N, f) \ : \ \begin{array}{c} N, \text{ 3-manifold with a complete hyperbolic metric} \\ f : \Sigma_g \to N \text{ realizing a homotopy equivalence} \end{array} \right\} \Big/ \sim$$

The equivalence relation is given by $(N_1, f_1) \sim (N_2, f_2)$ if they satisfy the analogous definition as for Teichmüller spaces. Then N is a complete hyperbolic 3-manifold with fundamental group $\Gamma \cong \pi_1\Sigma_g$, so we know that $N = \mathbb{H}^3/\Gamma$ where $\Gamma \subset \mathrm{PSL}(2, \mathbb{C})$ is a discrete subgroup.

Claim 15.7. $\mathrm{AH}(\Sigma_g) \cong DF(\pi_1\Sigma_g, \mathrm{PSL}(2, \mathbb{C}))/\,\mathrm{PSL}(2, \mathbb{C})$.

Then we look at the action of ψ on $\mathrm{AH}(\Sigma_g)$ and find a fixed point which will correspond to a discrete, faithful, representation fixed up to conjugacy by the action of ψ By iterating the action of ψ on a starting representation ρ_0, we can generate a sequence $\{\rho_n\}$ of representations that will converge to the fixed point. However, this sequence may not be bounded in the space and may not converge. Here's ths solution: look for a compact subset of this space.

[18] What does AH stand for?

This is where the *quasi-fuchsian representations* come in. A representation $\eta : \Gamma \to \mathrm{PSL}(2,\mathbb{C})$ is quasi-fuchsian if its limit set is a quasi-circle.[19] Quasi-Fuchsian representations are also discrete and faithful and so

$$QF(\Sigma_g) \subset \mathrm{AH}(\Sigma_g).$$

This is the set we will compactify in order to look for a fixed point. This approach is motivated by the following observation. We can do a partial quotient $\mathbb{H}^3/\pi_1\Sigma_g$ to get something which is homeomorphic to $\Sigma_g \times \mathbb{R}$ and which has two topological ends. The domain of discontinuity on S^2_∞ must then consist of two disjoint sets with the limit set being a simple closed curve.

So, we will start with a quasi-Fuchsian (actually a bonafide Fuchsian) representation $\eta : \pi_1\Sigma_g \to \mathrm{PSL}(2,\mathbb{C})$, and we will look at the sequence $\{\psi^n(\eta)\}$ and see what representation this converges to.

Remark 15.8 (convergence of representations). Representations ρ_n converge to some other rep ρ if for all $g \in \pi_1(\Sigma_g)$, $\rho_n(g) \to \rho(g)$ (in some reasonable sense of convergence).

How do we get a handle on the quasi-Fuchsian representations? There is an identification

$$\Psi : (\Sigma_g) \to \mathcal{T}(\Sigma_g) \times \mathcal{T}(\overline{\Sigma}_g), \qquad \Gamma \mapsto \left(\Omega^+_\Gamma/\Gamma,\ \Omega^-_\Gamma/\Gamma\right)$$

where $\Omega^\pm_\Gamma$ is the positive/negative component of the domain of discontinuity on S^2_∞.[20] The Fuchsian representations are those with image in $\mathrm{PSL}(2,\mathbb{R}) \subset \mathrm{PSL}(2,\mathbb{C})$ and these induce $\Psi(\Gamma) = (X, \overline{X})$. That is, the complex structures on $\Omega^\pm_\Gamma$ are the same but with their orientation reversed.

Theorem 15.9 (Bers)**.** *Ψ is a homeomorphism*

Definition 15.10. Fix $Y \in \mathcal{T}(\overline{\Sigma_g})$. Then *Bers slice B_Y* is $B_Y := \Psi^{-1}(\mathcal{T}(\Sigma_g) \times Y)$. (i.e. all representations which give a complex structure Y on the repelling end.)

Theorem 15.11. *$\overline{B_Y}$ is compact in* $\mathrm{AH}(\Sigma_g)$.

proof idea. The compactness of $\overline{B_Y}$ comes from the following. Let $\rho \in \mathrm{AH}(\Sigma_g)$ and $\mathrm{Im}(\rho) = \Gamma < \mathrm{PSL}(2,\mathbb{C})$. Suppose $X = \Omega^+_\Gamma/\Gamma$ and $Y = \Omega^-_\Gamma/\Gamma$. Note that X and Y are homeomorphic to Σ_g and $\mathbb{H}^3/\Gamma$ is homotopic to Σ_g. For $\gamma \in \pi_1\Sigma_g$, we have geodesic

[19]Heuristically, this looks like a circle but maybe very wiggly.
[20]The space $\mathcal{T}(\overline{\Sigma}_g)$ is the space of *anti-holomorphic* structures on Σ_g analogous to $\mathcal{T}(\Sigma_g)$.

representatives of γ in each space X, Y, and $\mathbb{H}^3/\Gamma$. Then we have the following (non-obvious) relation.

$$\frac{4}{\mathrm{len}_{\mathbb{H}^3/\Gamma}(\gamma)} \geq \frac{1}{\mathrm{len}_X(\gamma)} + \frac{1}{\mathrm{len}_Y(\gamma)}$$

And so $\mathrm{len}_{\mathbb{H}^3/\Gamma}(\gamma)$ is upper-bounded by a constant. □

Theorem 15.12 (Double Limit theorem (Thurston)). *For all* $(X, Y) \in \mathcal{T}(\Sigma_g) \times \mathcal{T}(\overline{\Sigma_g})$*, the set*

$$\{\Psi^{-1}(\psi^n(X), \psi^{-n}(Y)) : n \in \mathbb{N}\}$$

lies in a compact subset of $AH(\Sigma_g)$*.*

All this gives the main statement we want to prove: $\psi^m(M_\eta) \to M_\psi$. Sullivan Rigidity ?.

I don't know what is happening at the end of this proof.

16 Lecture 16: Fibered Links and embedded surfaces

Question 7. *Given an arbitrary 3-manifold, when can we decide if it is a surface bundle over a circle?*

We know how to build a fiber bundle $F \to M \to S^1$ given a monodromy representation $\pi_1 S^1 \to \mathrm{Homeo}^+(F)$. This is the inverse problem: when is M a fiber bundle and how many ways are there to do it?

Preliminary question: Given a link $L \subset S^3$, when is L a fibered link? (i.e. when is $S^3 \setminus L$ fibered by Seifert surfaces of the link?)

Definition 16.1 (Seifert Surfaces). Let L be a link in S^3. Choose an orientation on the link components. A *Seifert surface* is a properly embedded oriented surface $F \subset S^3 \setminus L$ with $\partial F = L$.

Example 16.2. Recall the Hopf fibration $S^1 \to S^3 \to S^2$. Embed the Hopf link H into $S^3 \cong \mathbb{R}^3 \cup \{\infty\}$ such that one link component is the unit circle and the other is the z-axis. Then with the same map as from the Hopf fibration, we get a fibration of the Hopf link complement over the sphere with 2 punctures.

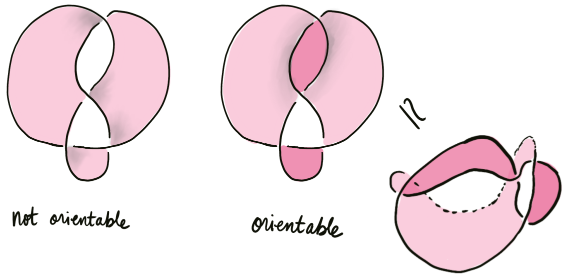

Figure 26: One way to construct a surface with the figure 8 knot as a boundary is to shade in alternate regions of the knot diagram. However, this produces a non-orientable surface. An orientable Seifert surface for the figure 8 knot is shown on the right.

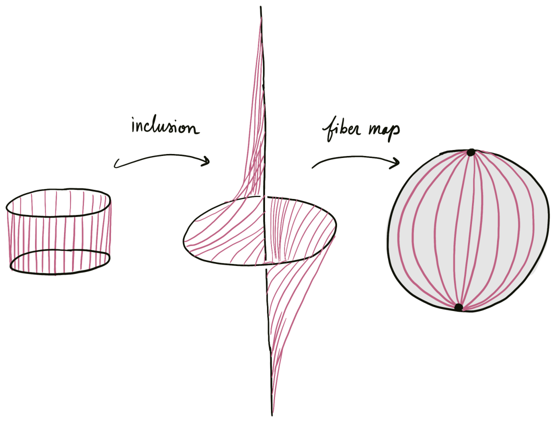

Figure 27: The Hopf link is fibered by its seifert surface which is an annulus.

$$\begin{array}{ccc} S^1 & \longrightarrow & S^3 \setminus H \\ & & \downarrow \\ & & S^2 \setminus \{0, \infty\} \end{array}$$

The pre-images of arcs from the north to south pole in S^2 are all annuli in the S^3 which are Seifert surfaces for H.

Let L be a fibered link and F be a Seifert surface of L such that we have a fibered link complement $F \to S^3 \setminus N(L) \to S^1$. Each copy of F maps to a point in S^1 and a neighborhood $N(F)$ maps to an interval. Thus, the complement $S^3/N(F)$ fibers over a contractible space and is therefore a trivial bundle. So $S^3 \setminus N(F) \cong F \times (0, 1)$ (see

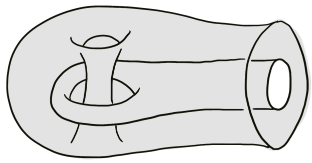

Figure 28: A 3-manifold with boundary of the form $F \times (0, 1)$.

figure 28). Note that the boundary of this 3-manifold is

$$\partial(F \times [0,1]) = (\partial F \times [0,1]) \cup (F \times \partial[0,1]).$$

Exercise: If F is $T^2 \setminus \{\text{disk}\}$, then boundary of $F \times (0,1)$ is Σ_2 and $F \times (0,1)$ is *isotopic* to a genus 2 handlebody!

16.1 Murasugi sum: *Combining two Seifert surfaces*

Let L_1 be a link with Seifert surface F_1 and let L_2 be a link with Seifert surface F_2. Let $P \subset F_1$ and $P \subset F_2$ each be a copy of the same even-sided polygon. Alternating sides of P should be embedded in the boundary of F_i. Then we can identify the two copies of P and "glue" the surfaces together to create a new link as the boundary. The resulting surface is the *Murasugi sum* of F_1 and F_2. If P is a bigon, then the new link will be the connected sum of the two links. We denote the resulting surface and link by $F_1 \#_P F_2$ and $L_1 \#_P L_2$ respectively. See Figure 29.

Theorem 16.3. *If L_1 and L_2 are fibered with fiber F_1 and F_2, then $L_1 \#_P L_2$ is fibered along $F_1 \#_P F_2$.*

Example 16.4 (From Hopf links to the Whitehead link). As discussed in Example 16.2, the Hopf link has an annulus as a Seifert surface. If we combine three such annuli with a Murasugi sum operation, we obtain a Seifert surface for the Whitehead link. See figure 30 for this operation.

Fact: Every *fibered* link is generated by doing and "undoing" Murasugi sums with a Hopf link!

Example 16.5. However, not all links are fibered. Note that the $(5, 2)$-torus knot has its complement is a genus 2 handle body but it is not fibered.

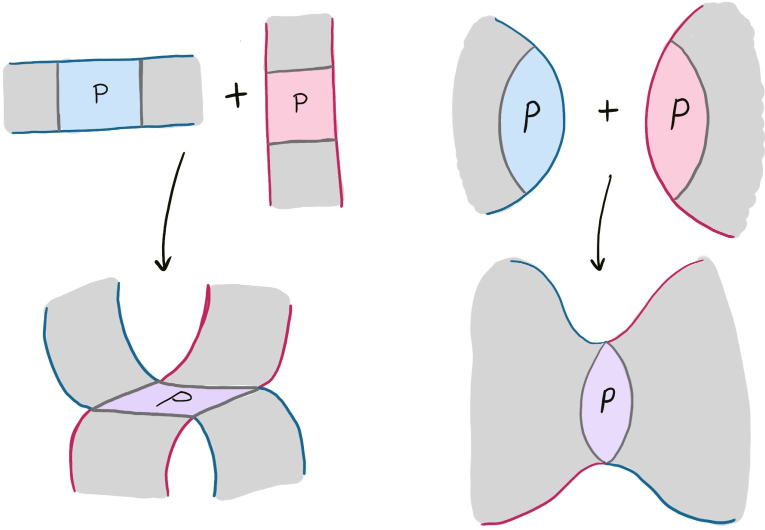

Figure 29: The local picture for a Murasugi sum with a 4-sided polygon (left) and a 2-sided polygon (right).

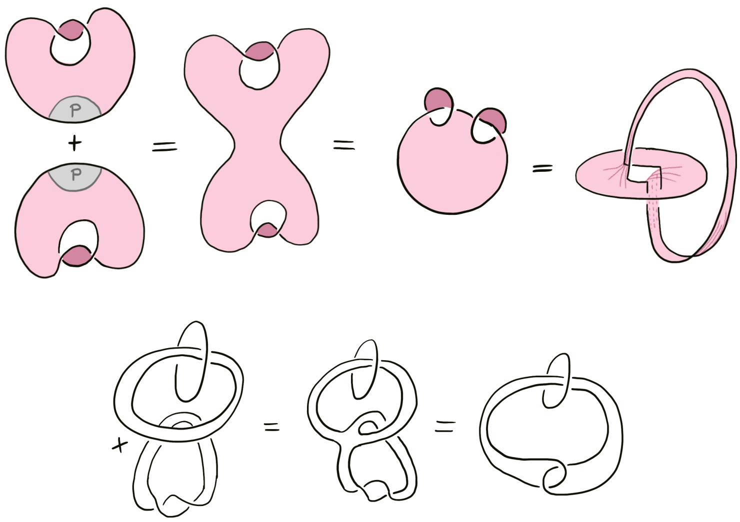

Figure 30: The Murasugi sum of two Hopf links along a bigon produces a Seifert surface for a 3-component link. Then, performing a Murasugi sum with a third Hopf link results in a Whitehead link.

16.2 The second homology of a 3-manifold

Let M^3 be compact, oriented, 3-manifold, irreducible, with incompressible boundary. (i.e. there is no embedded D^2 with its ∂ an essential curve in ∂M.[21]) Then $H_2(M, \partial M) = H^1(M)$ by Lefschetz duality and $H^1(M) = [M, S^1]$ where $[M, S^1]$ is the set of homotopy classes of maps $M \to S^1$.

Lemma 16.6. *Let S be a properly embedded 2-sided surface $(S, \partial S) \subset (M, \partial M)$ such that $[S] = \alpha \in H_2(M, \partial M)$. Then S is the pre-image of a regular value under a continuous map $f_\alpha : M \to S^1$ ($S = f_\alpha^{-1}(0)$).*

Proof. Look at the projection from a neighborhood of S to an interval.

$$\pi : S \times [-\varepsilon, \varepsilon] \to [-\varepsilon, \varepsilon]$$

Pick a 1-form α on $I = [-\varepsilon, \varepsilon]$ such that $\int_I \alpha = 1$ and $\alpha \to 0$ on ∂I. Then $\pi^*\alpha$ is a 1-form on $S \times I$. Extend $\pi^*\alpha$ by 0 to be a 1-form β on all of M. Then for a closed curve γ in M, $\int_\gamma \beta$ is the signed intersection number of γ with S. Thus, by definition, β is Poincaré dual to S.

To construct the map f_α, pick a basepoint $x_0 \in S$. Then for any $x \in M$, let γ_x be a path from x_0 to x. We define

$$f_\alpha(x) = \left(\int_{\gamma_x} \beta \right) / \mathbb{Z}$$

So f_α is a map $M \to \mathbb{R}/\mathbb{Z} = S^1$. The reader can check that f_α is well-defined (independent of path) and that $f_\alpha^{-1}(0) = S$. □

Corollary 16.7. *Let $\alpha \in H_2(M, \partial M; \mathbb{Z})$ be represented by properly embedded, two-sided surface $S \subset M$. If T is a properly embedded, two-sided surface representing $n\alpha$, then T is n parallel copies of S. That is, $T = S_1 \sqcup \cdots \sqcup S_n$, with each S_i isotopic to S.*

Proof. We find a smooth map $f : M \to S^1$ such that $f^{-1}(0) = S$. Then we can lift f to the $|n|$-fold cover of S^1 by the lifting criteria. If $\hat{f}$ is the lifted map, this implies that $\hat{f}^{-1}(0)$ consists of $|n|$ disjoint copies of S.

Exercise: Finish this proof.

□

[21] This is equivalent to saying that the inclusion of the boundary is π_1-injective

16.3 Thurston Norm

Remark 16.8. This was originally published in Thurston's paper *A norm for the homology of 3-manifolds.*

Suppose that M satisfies all the conditions as in the previous section. Let $S \hookrightarrow M$ be a two-sided properly embedded surface and let $\{S_i\}$ be the set of connected components of S which are not homeomorphic to $\mathbb{S}^2$ or D^2. We define

$$\chi^-(S) := \sum_i \chi(S_i)$$

Note that $[S] = [\cup_i S_i]$ in $H_2(M, \partial M; \mathbb{Z})$; this follows because M is irreducible with incompressible boundary so every embedded sphere is contractible and thus 0 in homology.

Lemma 16.9 (Embedded Surface)**.** *Let M be a compact oriented 3-manifold. Every class $\alpha \in H_2(M, \partial M; \mathbb{Z})$ is represented by $[S]$ where S is an oriented compact surface properly embedded in M.*

Lemma 16.9 allows us to define a map $\|\cdot\| : H_2(M, \partial M; \mathbb{Z}) \to \mathbb{R}$ given by

$$\|\alpha\| = \min\{-\chi^-(S) \mid [S] = \alpha\}.$$

This is the first step in defining the *Thurston norm* which is a norm on the vector space $H_2(M, \partial M; \mathbb{R})$.

Proposition 16.10 (Semi-norm)**.** *The map defined above on $H_2(M, \partial M; \mathbb{Z})$ is a* semi-norm. *That is, it satisfies the following properties.*

1. $\|\alpha\| \geq 0$ *for all* $\alpha \in H_2(M, \partial M; \mathbb{Z})$
2. $\|\alpha\| = \|-\alpha\|$
3. $\|n\alpha\| = |n|\|\alpha\|$ *for all* $k \in \mathbb{Z}$
4. $\|\alpha + \beta\| \leq \|\alpha\| + \|\beta\|$ *for all* $\alpha, \beta \in H_2(M, \partial M; \mathbb{Z})$

Proof sketch. –

1. By definition, $\chi^-(S) \leq 0$ for all S since all spherical and disk components have been excluded. Thus, $\|\alpha\| \geq 0$.
2. Euler characteristic is independent of orientation.

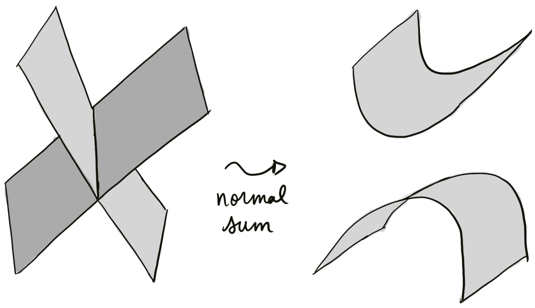

Figure 31: The normal sum of two surfaces intersecting generically.

3. Let S be a surface with $\|\alpha\| = -\chi^-(S)$. We can represent $k\alpha$ by $|k|$ parallel copies of S. This gives $\|k\alpha\| \leq |k|\|\alpha\|$. The other inequality follows from Corollary 16.7.

4. We represent α and β with Thurston-norm minimizing surfaces S and T. Compress any parts if they are compressible and perform a homotopy so we can assume they are incompressible surfaces which intersect in minimal position. Then we perform the normal sum (see Figure 31) along their intersection curves $T \cap S$. We will get no S^2 or D^2 components since they intersected in minimal position. This operation preserves total Euler characteristic and we obtain a properly embedded surface representing $[\alpha + \beta]$. This gives an upper bound on $\|\alpha + \beta\|$.

□

A manifold M is *atoroidal* if there are no essential tori in M (i.e. no embedded non-boundary parallel incompressible tori). A manifold with boundary is *acylindrical* if its double is atoroidal.

Proposition 16.11. *If M is atoroidal (and acylindrical if $\partial M \neq \emptyset$), then $\|\cdot\|$ is positive definite so it is a norm. That is, $\|\alpha\| = 0$ if and only if $\alpha = [0]$ in homology.*

Proof for M without boundary. Suppose that $\|\alpha\| = 0$ and α is represented by an incompressible surface S with $\|\alpha\| = -\chi^-(S)$. Let $\{S_i\}$ be the non-spherical connected components of S so that $\chi^-(S) = \sum_i \chi(S_i) = 0$. Note that $\chi(S_i) \leq 0$ for each component S_i since S_i is not $\mathbb{S}^2$. Hence, we conclude that $\chi(S_i) = 0$ for all components S_i. Since S is orientable, all components must be tori and S is the disjoint union of tori. However, M is atoroidal so S must be compressible. Thus, $\alpha = [0]$. □

We extend $\|\cdot\|$ to $H_2(M, \partial M; \mathbb{R})$ by first extending to homology classes with rational coefficients by using Proposition 16.10 part (2). Then we use the convexity in Proposition 16.10 part (3) and limiting sequences to define $\|\alpha\|$ for all $\alpha \in H_2(M, \partial M; \mathbb{R})$.

17 Lecture 17: Thurston Norm

Recall: We defined the Thurston norm on $H_2(M, \partial M; \mathbb{R})$ for M compact, oriented, irreducible, with incompressible boundary. This is a pseudo-norm under these conditions. If we insist that M is additionally atoroidal and acylindrical, then this is an honest norm.

Remark 17.1. Note that $H_2(M, \partial M; \mathbb{Z}) \subset H_2(M, \partial M; \mathbb{R})$ embeds as a lattice.[22] Then we extend the norm by linearity.

We then observe that $\|\cdot\|$ takes integer values on $H_2(M, \partial M; \mathbb{Z})$. If M is a closed 3-manifold, then it in fact takes *even* values because properly embedded surfaces will have no boundary components.

Proposition 17.2. *A norm $\|\cdot\|$ on $\mathbb{R}^n$ taking $\mathbb{Z}$ values on $\mathbb{Z}^n$ has a unit ball which is a finite-sided polyhedron.*

Remark 17.3. This is not true for the rationals. There are norms on $\mathbb{R}^n$ taking $\mathbb{Q}$ values on $\mathbb{Q}^n$ which are smooth. See if you can find one!

proof idea for Proposition 17.2. We give a brief idea of the proof in the case that $H_2(M, \partial M; \mathbb{R})$ is of rank 2.

Let $v_1 = (1, 0)$ and $v_2 = (0, 1)$ be a chosen basis for $\mathbb{R}^2$. Let $\|v_2\| = m$ and define $\overline{v_2} = \frac{v_2}{\|v_2\|} = (0, \frac{1}{m})$. Consider a sequence of vectors given by $w_k = v_1 + kv_2$ and define $\overline{w_k} = \frac{w_k}{\|w_k\|}$. Let ℓ_k be the line passing through $\overline{v_2}$ and $\overline{w_k}$. Note that as $k \to \infty$, $\overline{w_k}$ approaches $\overline{v_2}$ and ℓ_k will approach a line tangent to the unit ball $B_{\|\cdot\|}$.

We consider the slope of the line ℓ_k. Let $\|w_k\| = n_k$ and then $\overline{w_k} = \frac{1}{n_k}(1, k) = (\frac{1}{n_k}, \frac{k}{n_k})$. Then we calculate that the slope of ℓ_k is $\frac{mk - n_k}{m}$. In particular, this is a rational number with denominator m.

Now, we note that the slope of ℓ_k is bounded between the slope of ℓ_1 and the slope of the tangent line to $B_{\|\cdot\|}$ at v_2.[23] There are only finitely many rational numbers

[22] $H_2(M, \partial M; \mathbb{Z})$ will have no torsion if we insist that M is atoroidal.

[23] Even if v_2 is a vertex of $B_{\|\cdot\|}$ and there is no well-defined tangent line, there will still be a "right-hand limit" of tangent lines. This will not have infinite slope because $B_{\|\cdot\|}$ is symmetric about the origin.

with denominator m in any bounded interval, so there are only finitely many slopes possible. Thus, for large enough k, the line ℓ_k will actually be tangent to $B_{\|\cdot\|}$ and this means that it describes a face of the unit ball. $\square$

Example 17.4 (Whitehead link). Let W be the whitehead link with link components ℓ_1 and ℓ_2 and let $M = S^3 \setminus W$. Then we note that

$$H_2(M, \partial M; \mathbb{Z}) \cong H^1(M; \mathbb{Z}) \cong \mathrm{Hom}\left(H_1(M; \mathbb{Z}), \mathbb{Z}\right).$$

Note that $H_1(M; \mathbb{Z})$ is generated by the homology classes of two curves γ_1 and γ_2 linking with ℓ_1 and ℓ_2 respectively. So $H_1(M; \mathbb{Z}) \cong \mathbb{Z}^2$ and $\mathrm{Hom}\left(H_1(M; \mathbb{Z}), \mathbb{Z}\right)$ is determined by where it sends these generators. This gives $H_2(M, \partial M; \mathbb{Z}) \cong \mathbb{Z}^2$ and it is generated by α_1 and α_2 where α_i is the class of surfaces with ℓ_i as a boundary.

We can find explicitly a surface $S_i \cong T^2 \setminus \{\text{disk}\}$ with ℓ_i as a boundary as shown in Figures 32a and 32b. We calculate

$$\|\alpha_i\| \leq -\chi(S_i) = 1.$$

There is no orientable surface with one boundary component and Euler characteristic 0; this means $\|\alpha_i\| \neq 0$. Hence, α_1 and α_2 both lie on the unit ball $B_{\|\cdot\|}$. Next consider the class $\alpha_1 + \alpha_2$. By convexity, we know

$$\|\alpha_1 + \alpha_2\| \leq \|\alpha_1\| + \|\alpha_2\| = 2.$$

A surface with two boundary components has an even Euler characteristic. So either $\|\alpha_1 + \alpha_2\| = 0$ and there is an annulus with $\ell_1 \cup \ell_2$ as a boundary, or $\|\alpha_1 + \alpha_2\| = 2$. This first case cannot happen. By applying this same argument to $\alpha_1 - \alpha_2$ and then scaling by $\frac{1}{2}$ we obtain four more points on $B_{\|\cdot\|}$. Since the unit ball is a finite sided convex polytope, we have determined it to be the square with vertices $\pm(1,0)$ and $\pm(0,1)$.

17.1 Fiberings

Lemma 17.5. *Suppose F is a fiber of $F \to M \to S^1$. Then F is the unique incompressible surface in its homology class up to isotopy.*

proof sketch. Suppose that F' is another incompressible surface in the homology class. Lift F and F' to the $\mathbb{Z}$ cover $(\mathbb{R} \times F)$. Project to the surface. The map $\widetilde{F'} \to F$ must be a covering map and must have degree 1. We then get a homotopy and can promote it to an isotopy.

Exercise: Fill in the details of this proof.

$\square$

(a) (b)

Figure 32: The complement of the Whitehead link with two embedded surfaces with minimal genus in their homology class. Both surfaces are $T^2 \setminus \{\text{disk}\}$.

Let $\mathcal{F}$ be the foliation of M by surface fibers. Then $T\mathcal{F}$ is a subbundle of TM and it has an associated *Euler class*. This is a cohomology class $e(T\mathcal{F}) \in H^2(M;\mathbb{Z})$. Let $\sigma : M \to T\mathcal{F}$ be a generic section of the tangent bundle to the foliation. Then $e(T\mathcal{F})[S]$ counts the number of zeros with sign of the vector field $\sigma(S)$ on S. This is equivalently the signed intersection number of a generic section of $T\mathcal{F}$ with the zero secction.

Claim 17.6. *If F is a fiber of the foliation, then $e(T\mathcal{F})[F] = -\chi(F)$. Then $-e$ is a member of $H^2(M;\mathbb{Z})$ whose value on $[F]$ is $\|F\|$.*

Lemma 17.7. *Let M fiber over the circle with fiber F and let e be the Euler class associated with this foliation. Then the associated cohomology class $-e$ is a vertex of the unit ball of the dual Thurston norm. Furthermore, the ray determined by $[F]$ in $H_2(M;\mathbb{R})$ projectively intersects the interior of a top dimensional face of the unit ball $B_{\|\cdot\|}$.*

Proof. First, we establish the following inequality. If S is any closed oriented surface and $f : S \to M$ is a map, then $|e(f(S))| \leq \|S\|$. A proof of this is given in [2] and then rephrased in [3]. It follows from a fact that the Euler class will count singularities of a vector field with sign while the Euler characteristic counts them without (à la Poincaré-Hopf). These two counts will be equal when we have only saddle singularities of the vector field, which happens exactly when the surface is parallel to a fiber.

Then we recall the definition of the dual norm on a dual vector space.

$$\|e\|^* = \sup_{\alpha \in H_2(M;\mathbb{R})} \left\{ \frac{1}{\|\alpha\|} e(\alpha) \right\}$$

Then the inequality above and the fact that $-e(F) = \chi(F)$ establishes that e is on the boundary of the dual unit ball.

Exercise: Finish the last bit of this proof.

□

Theorem 17.8. *If $[F]$ is a fiber of a fibration, then $[F]$ projectively intersects the interior of a top dimensional face of the unit ball and is dual to $-e(T\mathcal{F})$. And every other integer class projectively intersecting this face is also a fiber.*

A face of the unit norm ball described in Theorem 17.8 is called a *fibered face*. The fibered faces are characterized by all the nowhere vanishing closed 1-forms on the 3-manifold. These are dual to the fibered faces and are represented by vertices in the unit ball of the dual norm.

Theorem 17.9. *A closed n-manifold M^n fibers over S^1 iff M has a nowhere 0 closed 1-form ω.*

A fibration over the circle is a foliation with *compact* leaves. There is a folklore idea that as we pass through the lower dimensional faces of the unit norm ball on $H_1(M, \partial M)$, we transition from one fibration to another, and we must "pass through" a foliation with non-compact leaves. In [4], the authors build up this theory using transverse flows on the 3-manifold. If M is a hyperbolic 3-manifold which fibers over the circle with compact fiber, then the surface bundle has *pseudo-Anosov monodromy*. The authors of [4] propose an analogous idea for non-compact surfaces call *spun pseudo-Anosov maps*.

17.2 Taut Foliations

Definition 17.10 (taut foliation)**.** A co-oriented foliation $\mathcal{F}$ of a 3-dimensional M by surfaces is *taut* if there is a closed oriented transversal intersecting every leaf of $\mathcal{F}$.

Example 17.11. The foliation $\mathcal{F}$ of a mapping torus M_f comprised of leaves $\Sigma_g \times \{t\}$ is a taut foliation. Take the path $x \times [0,1]$ for some $x \in \Sigma_g$. Then add in a path between the endpoints $(x, 0)$ and $(f(x), 0)$ lying in the leaf $\Sigma_g \times \{0\}$ to obtain a closed curve γ. We can perform a homotopy on γ to make it transverse to every leaf of $\mathcal{F}$. Since f is an orientation-preserving homeomorphism, we can assign a co-orientation to this transversal.

There are many different equivalent formulations of a taut foliation; each is useful in a different context. We list a few which are most relevant to us but a more extensive list can be found in [3].

Proposition 17.12 (Equivalent definitions of tautness)**.** *The following are equivalent conditions for $\mathcal{F}$ on M^3 being a taut foliation.*

1. *For every point $p \in M$, there is an immersed circle transverse to $\mathcal{F}$ passing through p.*
2. *There is no proper closed submanifold N of M whose boundary is tangent to $\mathcal{F}$ and for which the co-orientation points in to N along ∂N.*
3. *There is a closed 2-form ω on M positive on $T\mathcal{F}$.*
4. *There is a flow X transverse to $\mathcal{F}$ which is volume preserving for some Riemannian metric on M.*
5. *There is a Riemannian metric on M for which every leaf of $\mathcal{F}$ is a minimal surface.*
6. *There is a map $f : M \to \mathbb{S}^2$ whose restriction to each leaf λ is a branched covering.*

Proposition 17.13. *An embedded, oriented, closed surface $S \hookrightarrow M$ is Thurston norm minimizing if and only if it is the compact leaf of a taut foliation on M.*

Proof. ($\Leftarrow$) This follows from the inequality $|e(\mathcal{F})[S]| \leq |\chi^-(S)|$ for all embedded, oriented, closed surfaces S and the fact that equality is achieved when S is a compact leaf of $\mathcal{F}$.

($\Rightarrow$) This is a theorem of Gabai [5]. The proof proceeds by decomposing M along Thurston norm minimizing surfaces to obtain taut sutured manifold pieces. Since taut sutured manifolds have finite hierarchies, this procedure will terminate. Then the pieces of the decomposition can be foliated with non-compact leaves and the decomposing surfaces are compact leaves of the resulting foliation. □

Example 17.14. Let $K \subset S^3$ be any knot and S be any Seifert surface of this knot. Then take $S^3 - N(S)$ to be your sutured manifold. The interior of the manifold is foliated by copies of S which either intersect the boundary of $S^3 - N(S)$ transversely (around the torus neighborhood of K) or are contained in the boundary.

Theorem 17.15. *The Murasugi sum of two taut foliations is also taut. The Murasugi sum of two norm-minimizing surfaces is also norm-minimizing.*

The result of this theorem, along with sutured manifold theory, allows us to prove that the Murasugi sum of two fibered knot complements is also a fibered knot complement.

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