Leafwise branched coverings of foliated 3-manifolds

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- examples of foliated 3-manifolds
- taut and depth one foliations
- Leafwise branched coverings

Examples of foliated 3-manifolds:

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Mapping Tori !!

S surface, f homeo$^+$ on S.

$$\boxed{M_f := S \times [0,1] \big/ (x,1) \sim (f(x),0)}$$

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$$\boxed{M_f := \frac{S \times [0,1]}{(x,1) \sim (f(x),0)}}$$

$\longrightarrow$ foliation on M_f is $\{S \times t\}$

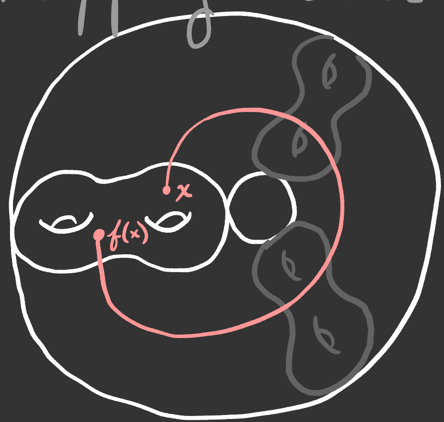
Examples of foliated 3-manifolds:

Σ_g finite surface

$f \in MCG(\Sigma_g)$

M_f

Mapping torus



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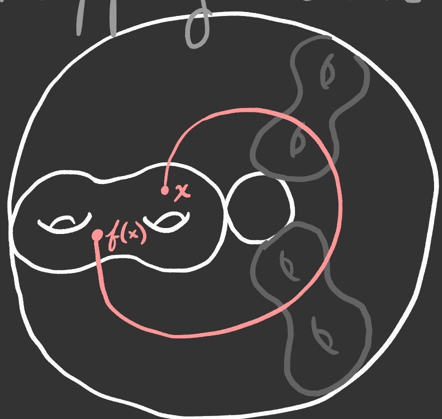
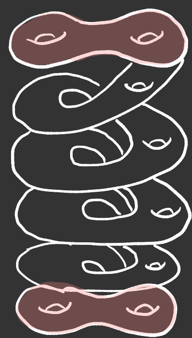


S infinite-type surface

$f \in MCG(S)$ end-periodic

$\overline{M_f}$

compactified mapping torus



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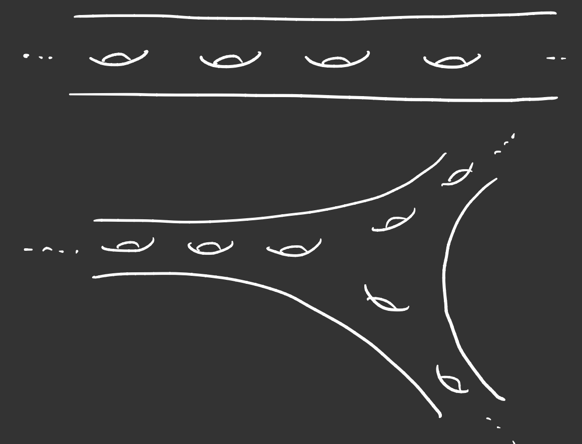

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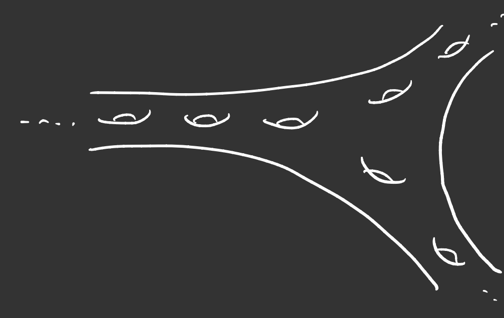
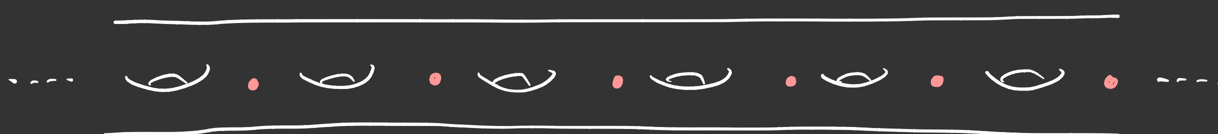
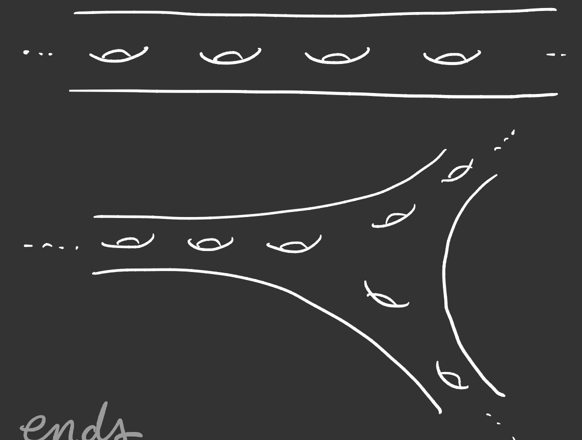




$\xrightarrow[\sigma]{}$ "shift map"

End-periodic mapping tori



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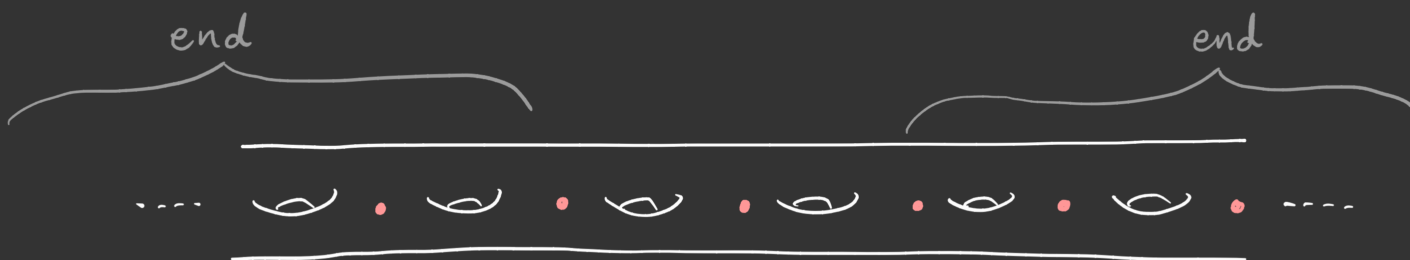
↳ "looks like" σ on the ends



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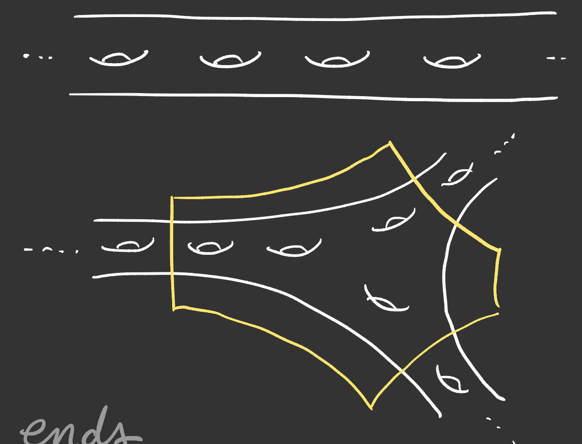

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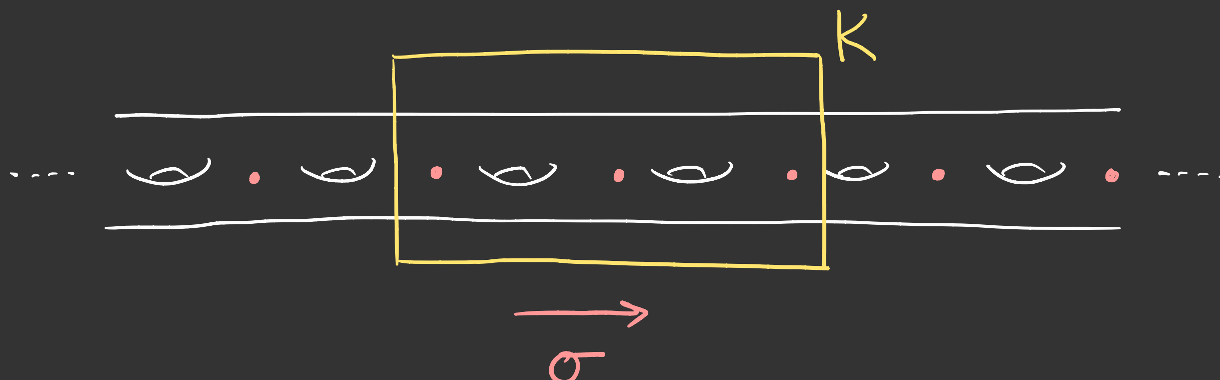

$f = \sigma \circ g$ supported in K

End-periodic mapping tori

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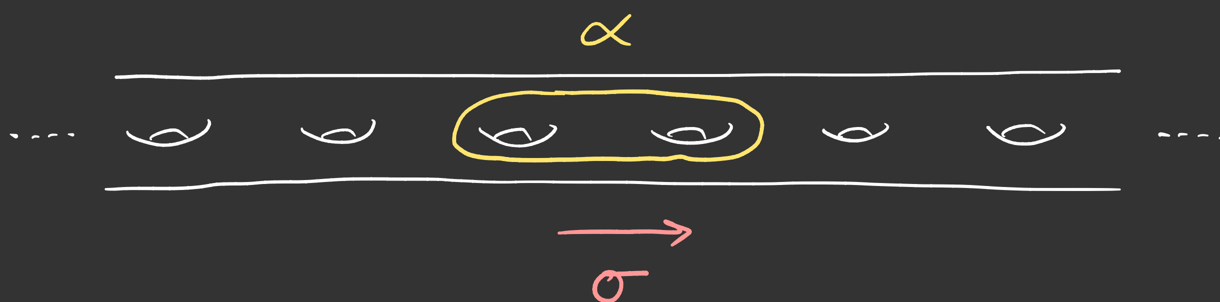
Ex: $f = \sigma \circ T_\alpha$



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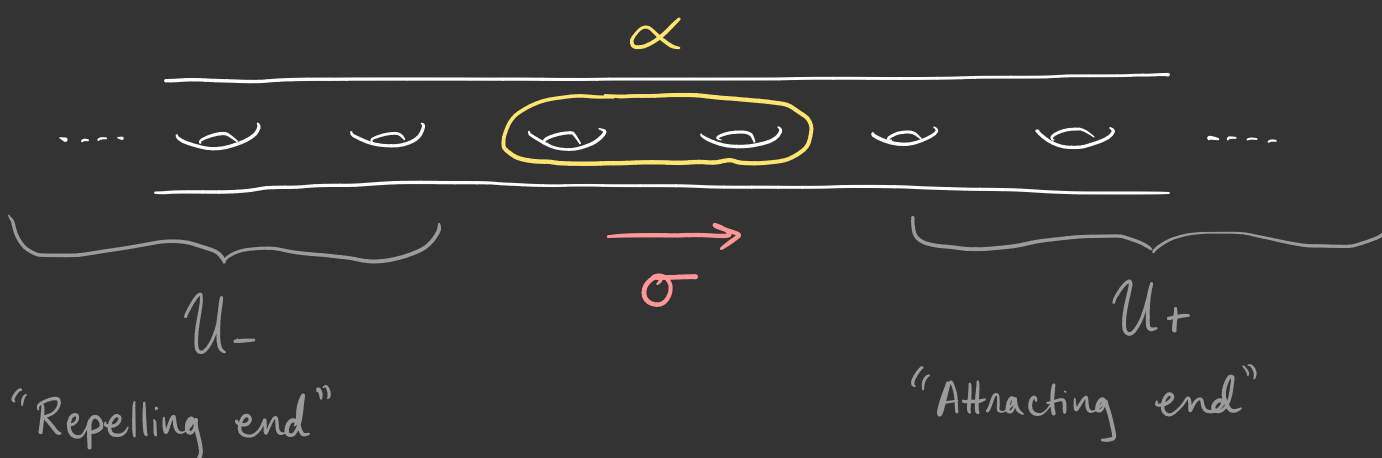

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M_f mapping torus

$$\boxed{M_f = \frac{S \times [0,1]}{(x,1) \sim (f(x),0)}}$$

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COMPACTIFY!

$$\boxed{\overline{M_f} = M_f \cup \mathcal{U}_+ / \langle f \rangle \cup \mathcal{U}_- / \langle f \rangle}$$

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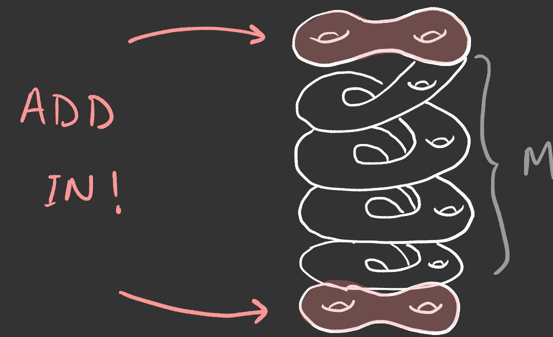

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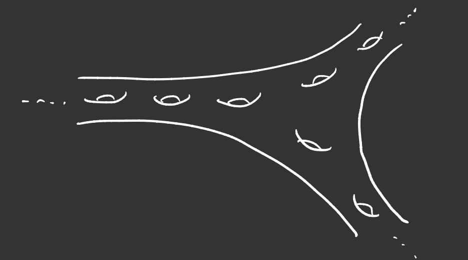

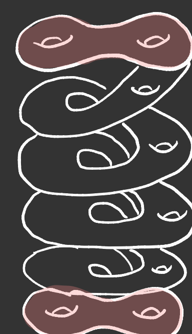

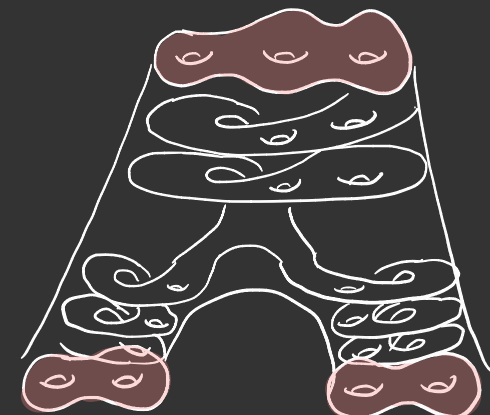

Leafwise branched coverings of foliated 3-manifolds

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Definition ??

leaves

Foliation : M 3-mnfld. $\mathcal{F} = \{\lambda \mid \lambda \subset M$ disjoint embedded surfaces$\}$ such that $M = \bigsqcup \lambda$ and locally $\mathbb{R}^2 \times \mathbb{R}$ foliated as a product.

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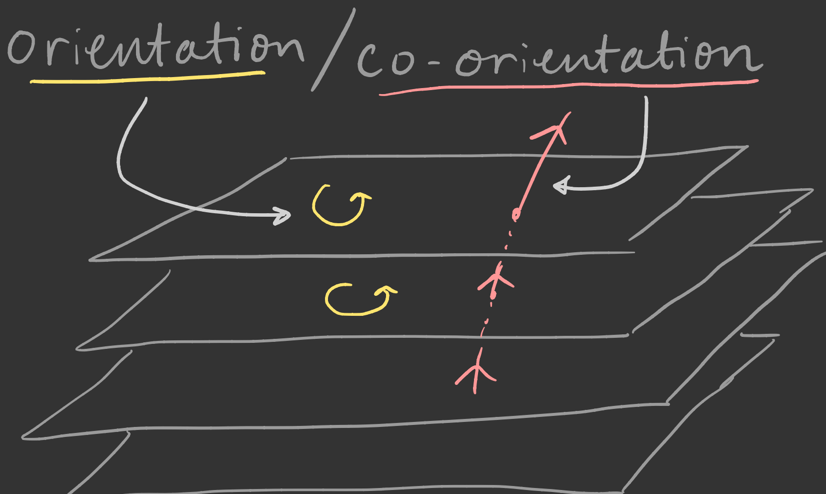

oriented

transversal

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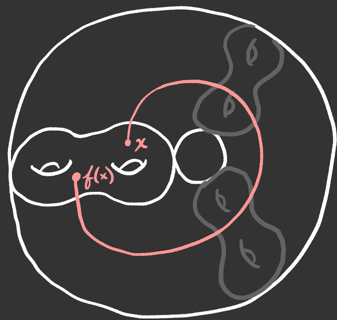


depth zero

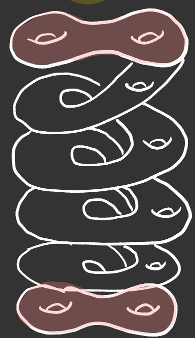
all leaves compact

S infinite-type

$f \in MCG(S)$

end-periodic

$\overline{M_f}$



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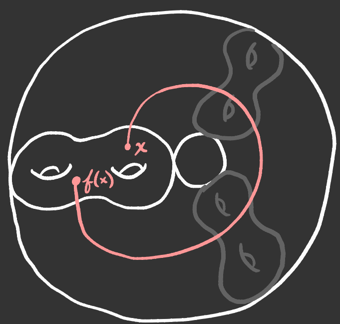

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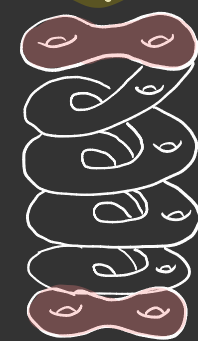

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FACT: all depth one foliations are made of pieces $\overline{M_f}$.

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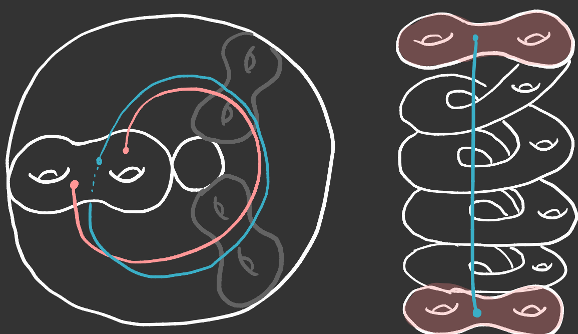
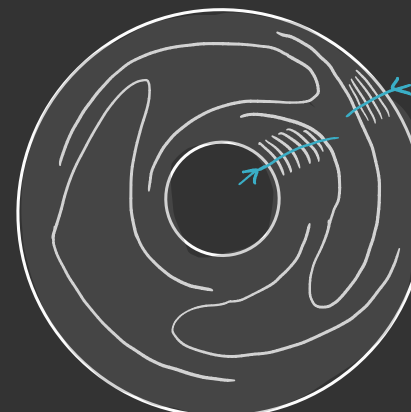
TAUT



NOT TAUT



[Reeb Component]

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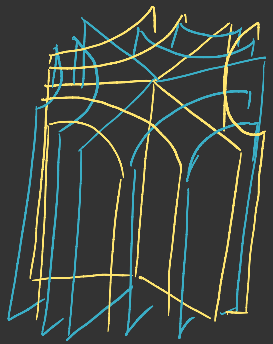

pseudo-Anosov flows transverse to $\mathcal{F}$!

(Sam Taylor, Michael Landry, Yair Minsky)

$\longrightarrow$ UNIVERSAL CIRCLES!

representation

$\rho: \pi_1(M) \longrightarrow Homeo^+(S^1)$

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3. There is a continuous map $\psi : M \to S^2$ such that for each $\lambda \in \mathcal{F}$, the restriction $\psi|_\lambda$ is a branched covering. [Calegari, Ghys, Donaldson]

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"Leafwise Branched Covering"

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YAY

Branched Covering Maps

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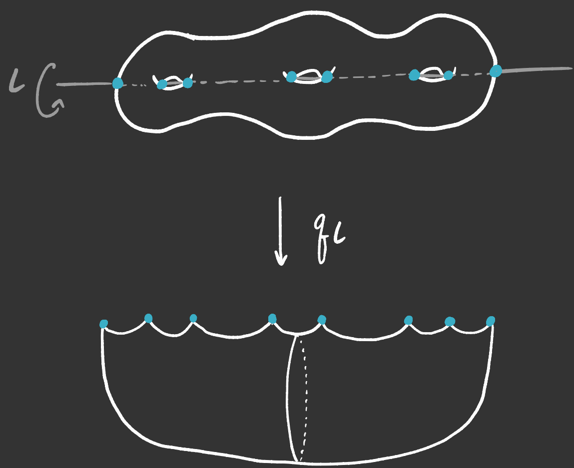
1. quotient by a finite order homeo



[hyperelliptic involution]

Branched Covering Maps

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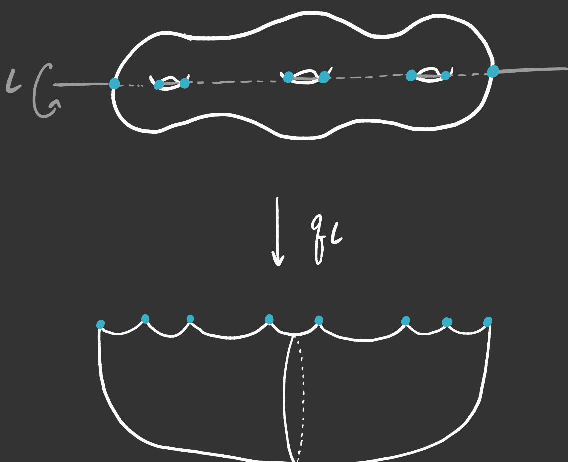

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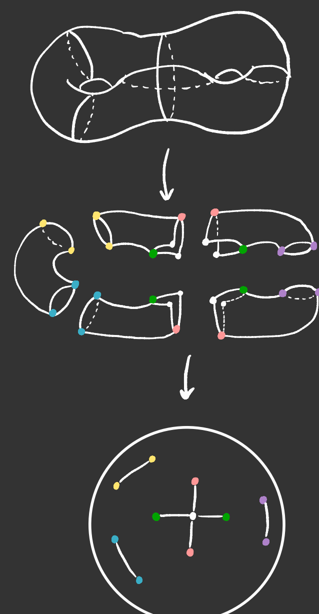

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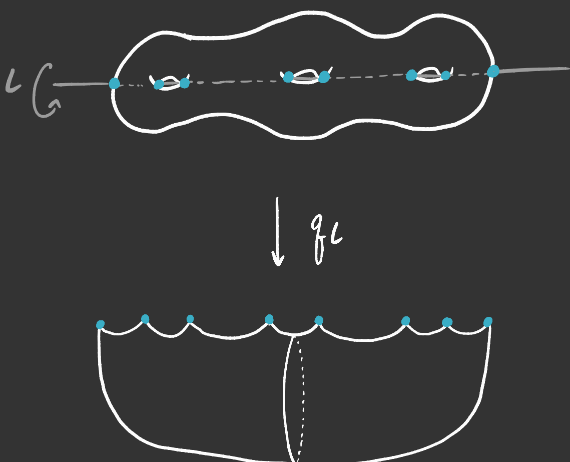

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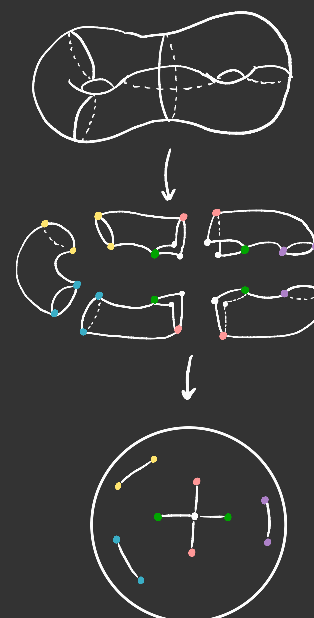

3. Rational Function

$$\hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$$

$$z \longmapsto \frac{z^3 - 1}{z + 2}$$

{Riemann Surfaces as Algebraic Curves}

Branched Covering Maps

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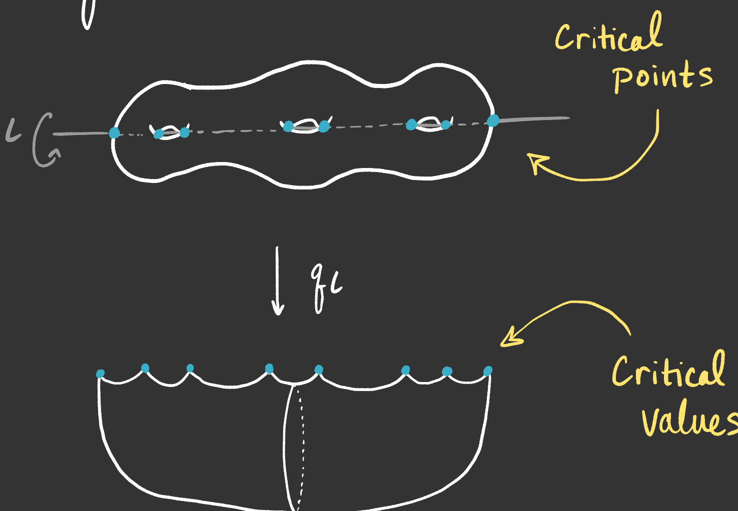

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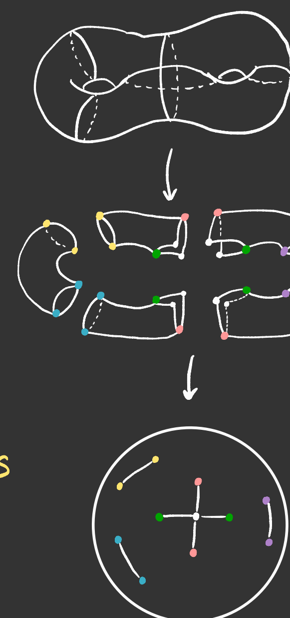

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{ Riemann Surfaces as Algebraic Curves }

Leafwise branched coverings on mapping tori

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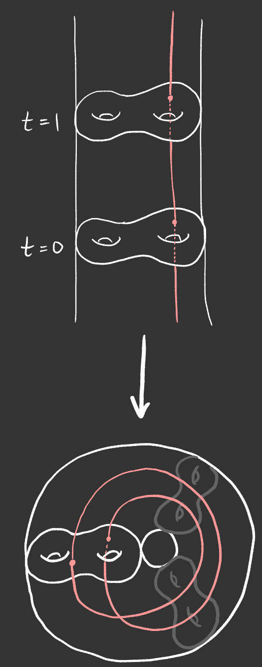

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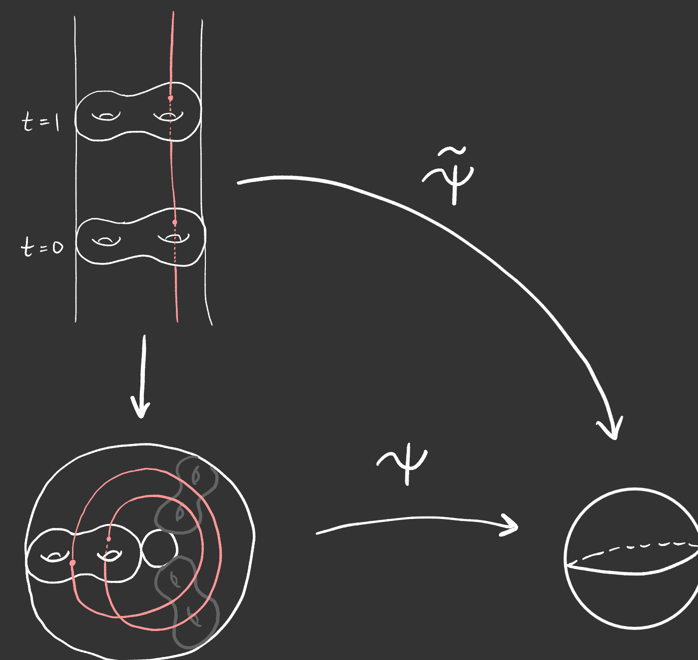
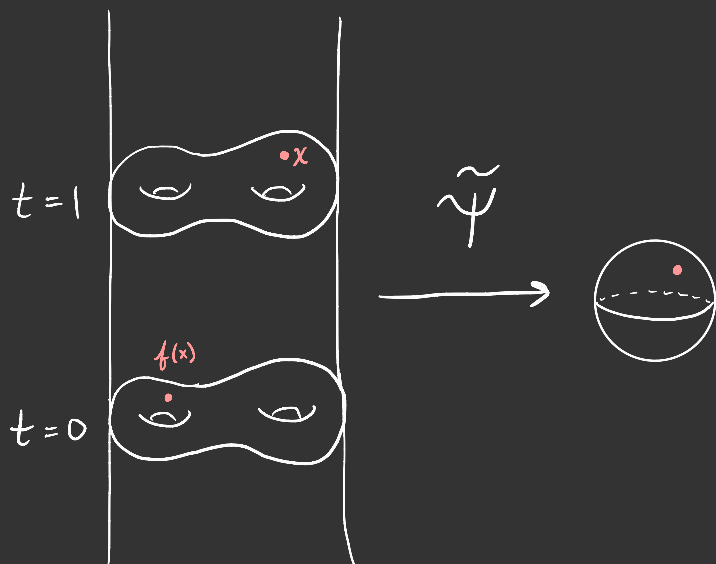


Leafwise branched coverings on mapping tori

Start with $\tilde{\Psi} : S \times \mathbb{R} \longrightarrow S^2$



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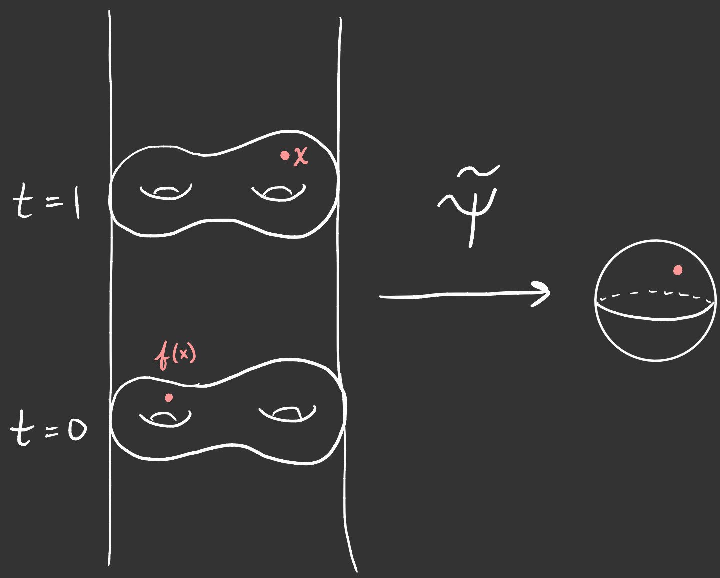
Start with $\tilde{\Psi} : S \times \mathbb{R} \longrightarrow S^2$

Q: When does $\tilde{\Psi}$ descend to M_f?



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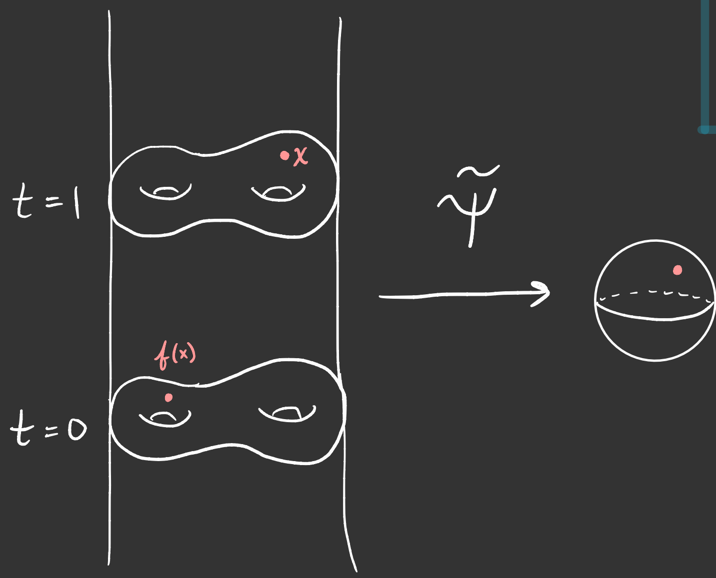

NEED: $\tilde{\Psi}(x,1) = \tilde{\Psi}(f(x),0)$

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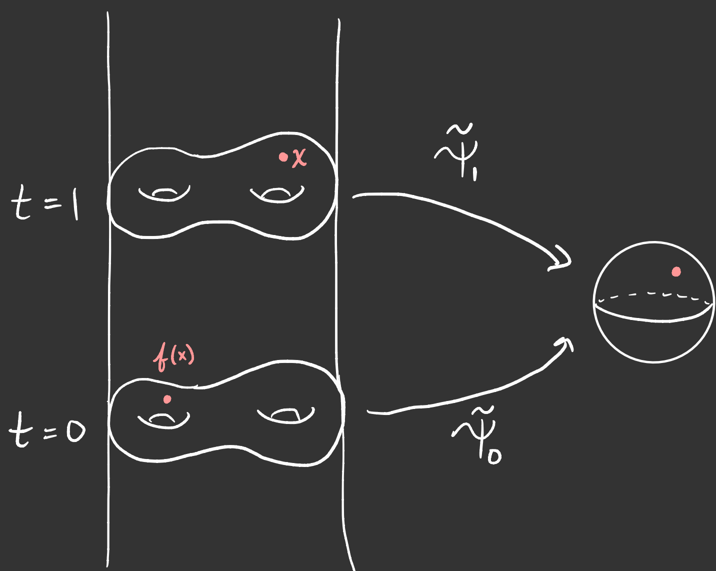

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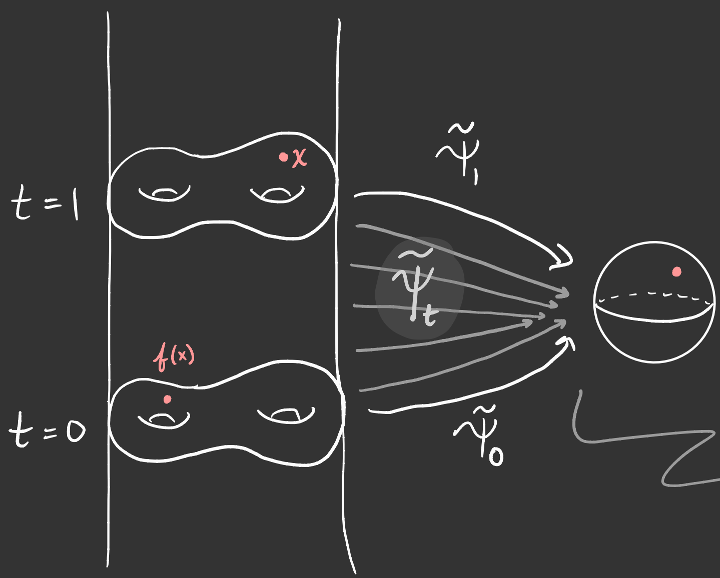

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$\tilde{\Psi}_0 \simeq \tilde{\Psi}_1$

"homotopy through branched coverings"

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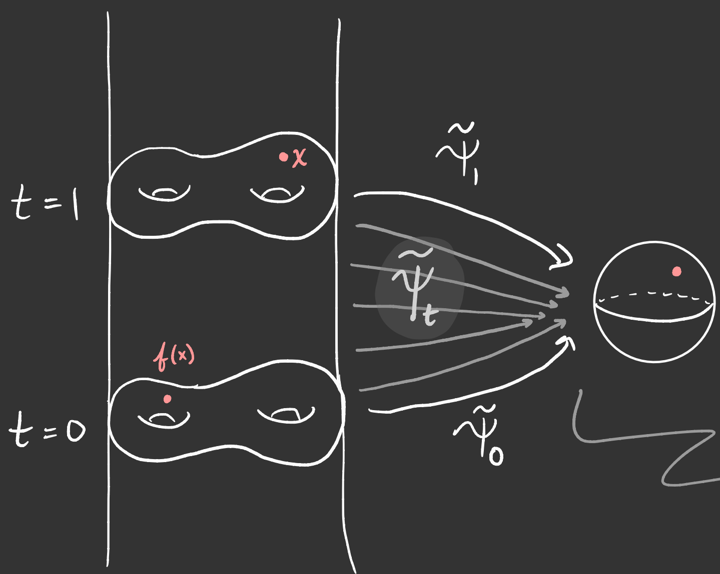

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$\tilde{\Psi}_0 \simeq \tilde{\Psi}_1 = \tilde{\Psi}_0 \circ f$

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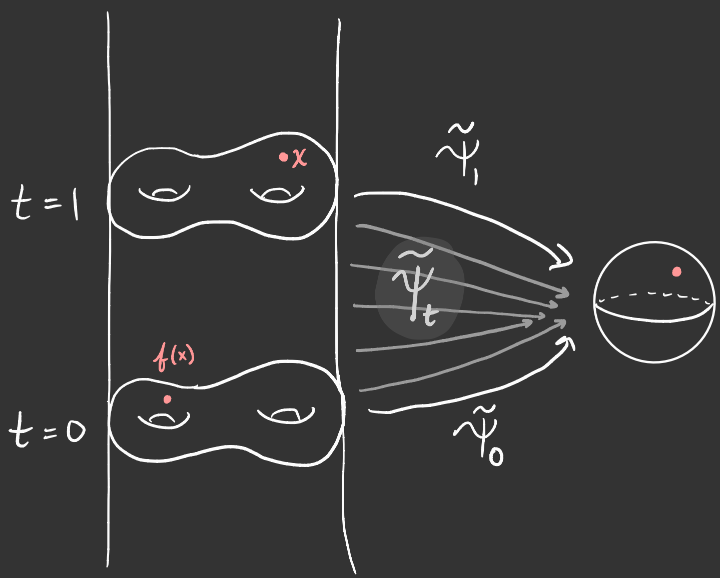

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NEED : $\boxed{\tilde{\Psi}_0 \simeq \tilde{\Psi}_0 \circ f}$

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find Ψ_0 and f such that

$$\Psi_0 \simeq \Psi_0 \circ f$$

Examples: $\Psi \simeq \Psi \circ f$

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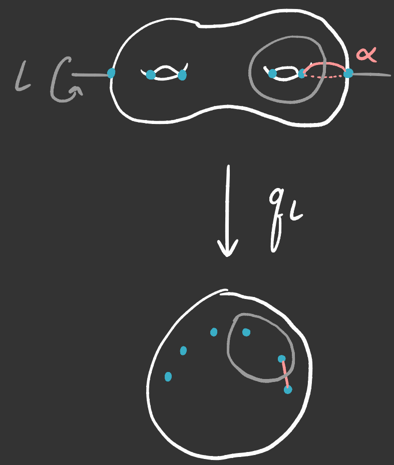
1 Hyperelliptic Homeomorphisms

$$f \circ \iota = \iota \circ f$$



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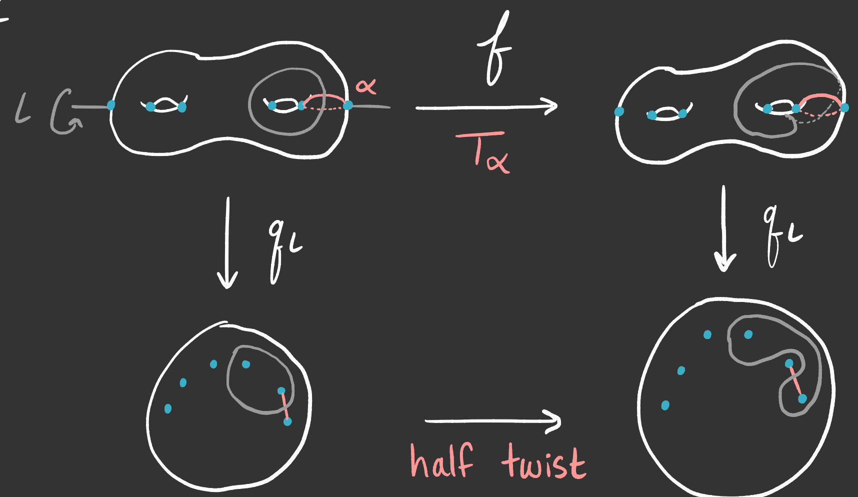
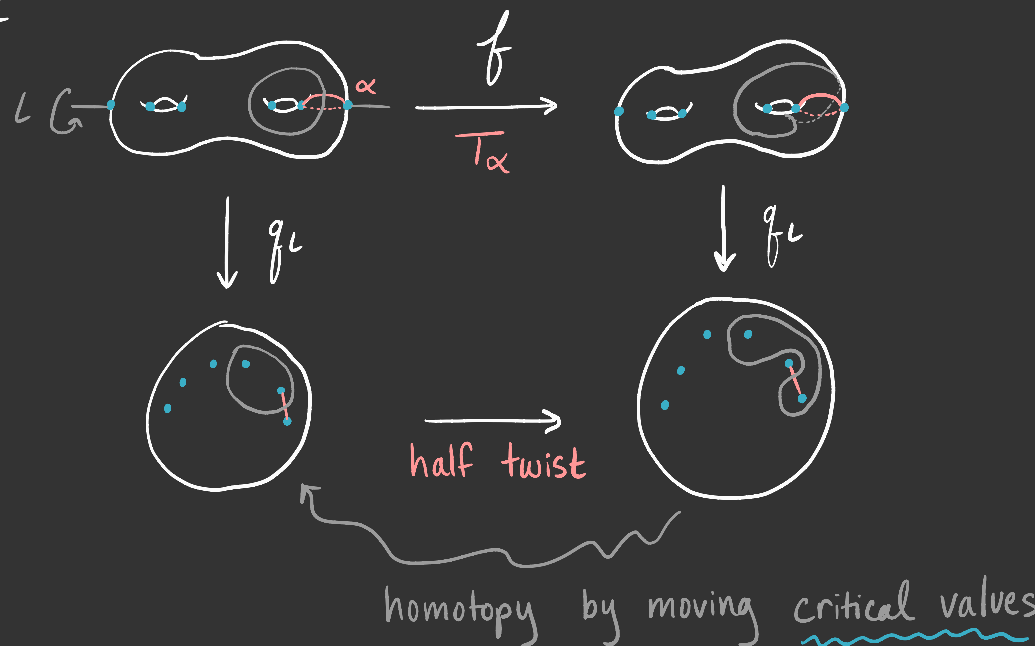


Examples: $\psi \simeq \psi \circ f$

① Hyperelliptic Homeomorphisms

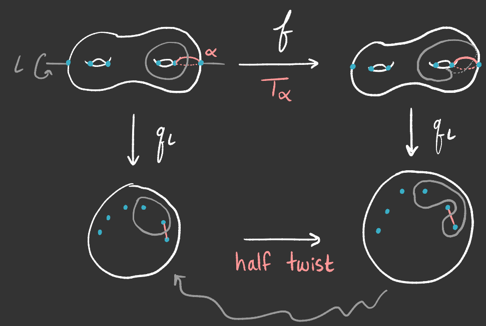
$f \circ \iota = \iota \circ f$



Examples: $\Psi \simeq \Psi \circ f$

① Hyperelliptic Homeomorphisms $\longrightarrow \Psi = q_\iota$

$$f \circ \iota = \iota \circ f$$



Examples: $\Psi \simeq \Psi \circ f$

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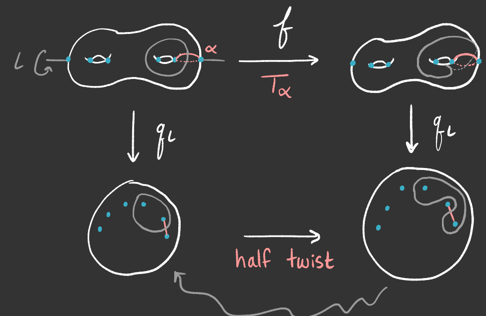


② Commuting Homeos $\longrightarrow$ $\Psi = q_\varphi$

φ finite order, $\quad f \circ \varphi = \varphi \circ f$

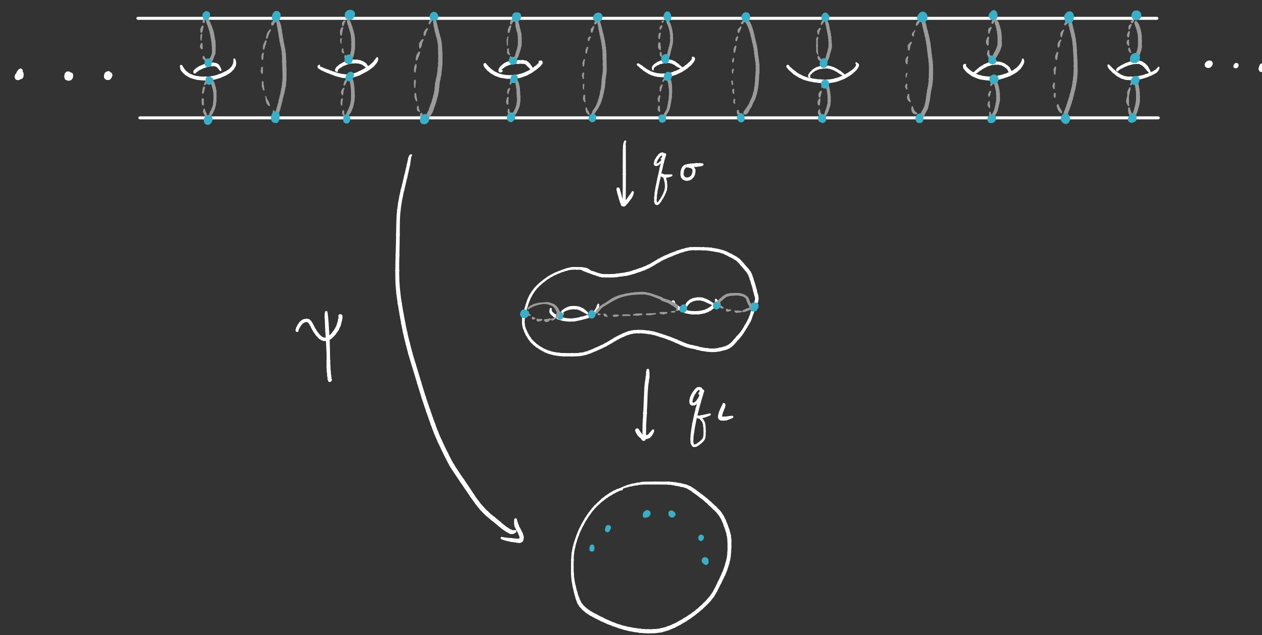
$[\varphi^k = \mathrm{id}]$

Examples: $\psi \simeq \psi \circ f$

3. Infinite-type surfaces ??

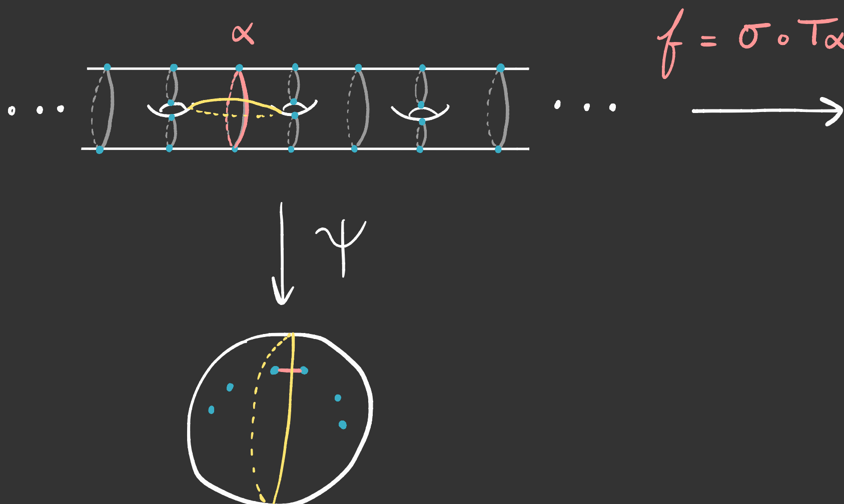
Examples: $\psi \simeq \psi \circ f$

③ Infinite-type surfaces



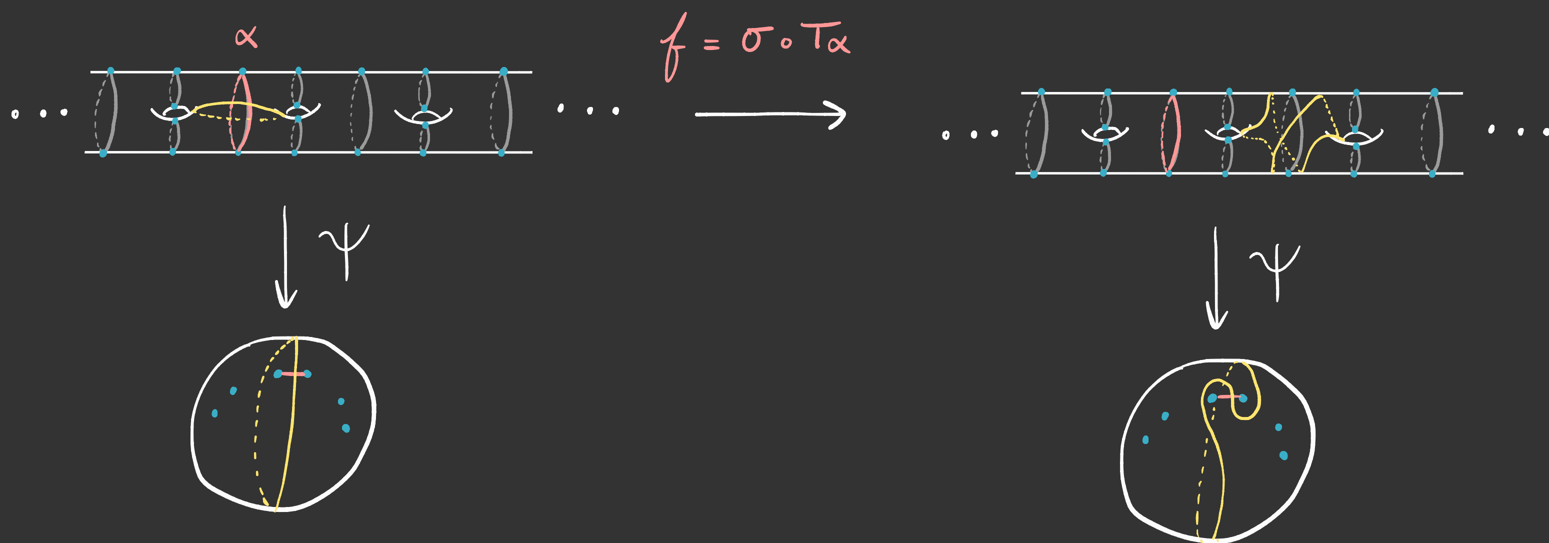
Examples: $\psi \simeq \psi \circ f$

③ Infinite-type surfaces



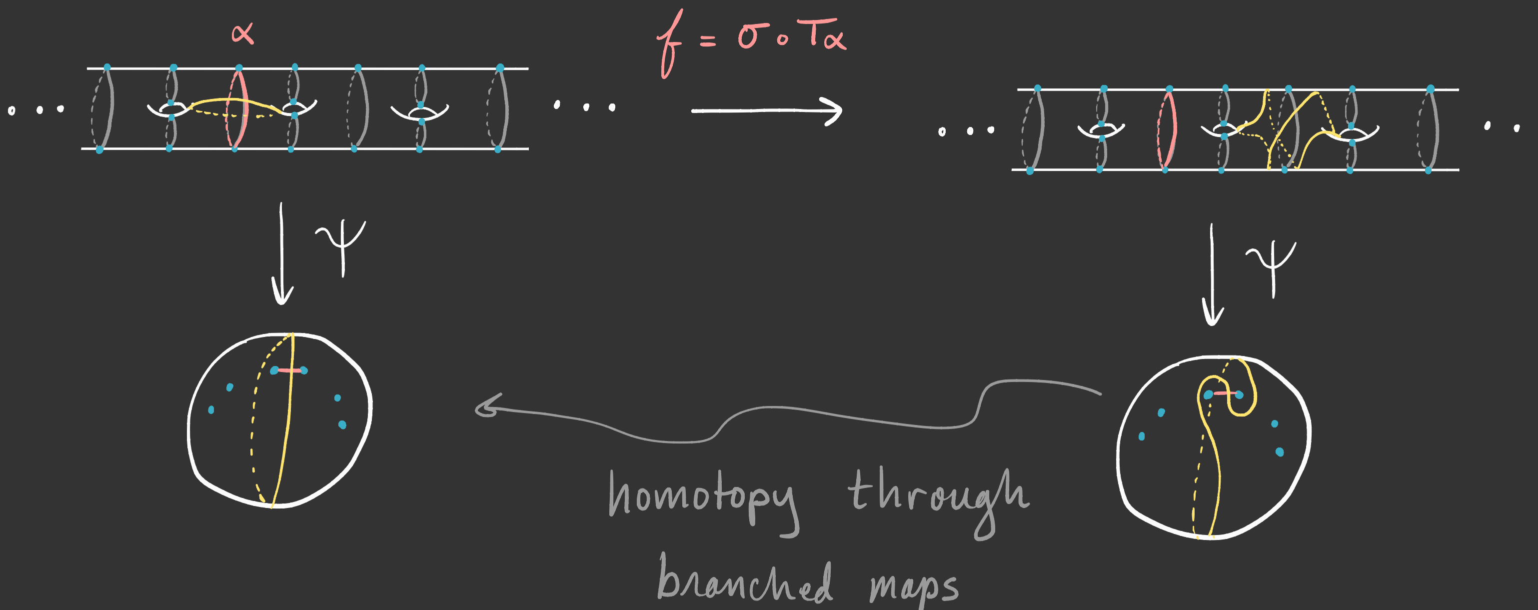
Examples: $\psi \simeq \psi \circ f$

3 Infinite-type surfaces



Examples: $\Psi \simeq \Psi \circ f$

3 Infinite - type surfaces



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$\boxed{\Psi \simeq \Psi \circ f}$ $\longrightarrow$ $\boxed{\text{Leafwise Branched Cover on } M_f \text{ or } \overline{M_f}}$

"Time to do some math!"

— a student of mine

THANK YOU!