Unpacking Thurston's Theorem for post-critically finite maps

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1 Introduction

Our main object of study will be *branched covering maps* of the sphere. For our purposes, $f : \mathbb{S}^2 \to \mathbb{S}^2$ is a branched covering map if it is a covering map except at finitely many points, which we will call the set of critical points $\Omega_f \subset \mathbb{S}^2$. So the map

$$f|_{\mathbb{S}^2 \setminus \Omega_f} : \mathbb{S}^2 \setminus \Omega_f \to \mathbb{S}^2 \setminus f(\Omega_f)$$

is a covering map in the normal sense. We will only be considering finite-sheeted branched covering maps. If the restriction $f|_{\mathbb{S}^2 \setminus \Omega_f}$ is a d-sheeted covering map, then for every critical point $x \in \Omega_f$, the fiber $f^{-1}(x)$ contains fewer than d points.

Definition 1.1. The *degree of f at x* is defined as follows: let U be a sufficiently small neighborhood of x so that $U \setminus \{x\}$ contains no critical points. Then, f induces a map on the quotient

$$f_U : \mathbb{S}^2/(\mathbb{S}^2 \setminus U) \to \mathbb{S}^2/\left(\mathbb{S}^2 \setminus f(U)\right).$$

And $\deg_x f$ is the degree of f_U as a map on spheres in the topological sense.

Heuristically, $\deg_x f$ is the integer k so that on a neighborhood of x, f looks like $z \mapsto z^k$. One rich source for examples of branched covering maps is the quotient by a group action which is not free. For example, consider $C_2 = \mathbb{Z}/2\mathbb{Z}$ acting on $\mathbb{S}^2$ by rotation about the axis through the north pole (N) and the south pole (S). This action is not free because N and S are fixed points of the non-identity rotation. But $\mathbb{S}^2 \to \mathbb{S}^2/C_2 \cong \mathbb{S}^2$ is a continuous map of degree 2. Also, when we identify $\mathbb{S}^2$ with the Riemann sphere $\mathbb{C} \cup \{\infty\} = \mathbb{P}^1$, then we can realize this topological map as the polynomial $z \mapsto z^2$. This example is shown in Figure 1a.

For another example, consider the dihedral group D_3 acting on the sphere. This action is generated by two rotations. Let the first be a $\frac{1}{3}$ rotation about the north/south pole axis; call this α. And let the second be a $\frac{1}{2}$ rotation about an axis through the equator; call this β. So

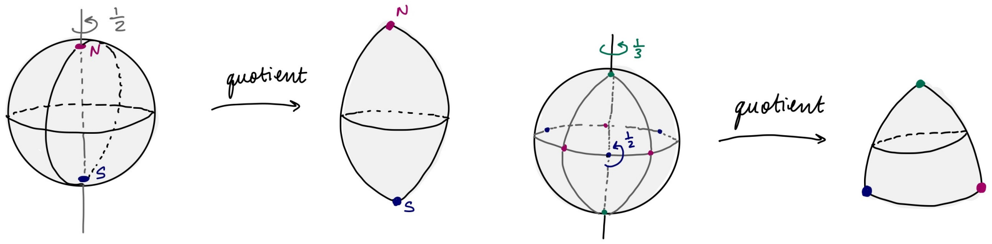
$$D_3 = \langle \alpha, \beta \mid \alpha^3 = \beta^2 = 1 \rangle.$$

Figure 1b shows an example of this. Then $\mathbb{S}^2/D_3$ is a sphere with three cone points. We can also realize this quotient map analytically. Let $\varphi_\alpha : z \mapsto z^3$. Then we need a Möbius transform which sends $(i, -i)$ to $(0, \infty)$. The map

$$T(z) = \frac{z - i}{z + i}$$

will work. And lastly, let $\varphi_\beta : z \mapsto z^2$. Then we can make the composite map

$$\phi(z) = (T^{-1} \circ \varphi_\beta \circ T \circ \varphi_\beta)(z).$$



(a) A 2-sheeted branched covering map with two critical points of degree 2. This is equivalent to the polynomial map $z \mapsto z^2$.

(b) A 6-sheeted branched covering map with 8 critical points: two with degree 3 and six with degree 2.

Figure 1: Two examples of topological brached covering maps of the sphere.

The map ϕ will make the same identifications as the quotient map $\mathbb{S}^2 \to \mathbb{S}^2/D_3$. Also, ϕ is the composition of rational maps, so it is also rational.

In these examples, we were able to start with a topological map f on the sphere and then construct a rational map on $\mathbb{P}^1$ which is "equivalent" to f. We would like our notion of equivalence to mean that f acts on $\mathbb{S}^2$ in the same way as the rational map; we can do this by considering the dynamics of f under iteration. The main question for this paper is to characterize for which topological maps can we find such a rational function.

Question 1. *Given a topologial branched covering $f : \mathbb{S}^2 \to \mathbb{S}^2$, when can we find a rational map $g : \mathbb{P}^1 \to \mathbb{P}^1$ so that f and g are equivalent?*

This was a question asked by Thurston in the 1980's. He stated and proved his answer– Theorem 2.6– throughout 1982-1983 on several occasions. Douady and Hubbard first reconstructed Thurston's argument and published it in their paper "A proof of Thurston's topological characterization of rational functions" in 1993 [1]. More recently, Hubbard has fleshed out the proof and added more exposition in the second volume of his book series on Teichmüller Theory: *Surface Homeomorphisms and Rational Functions* [2]. All of the proofs and framework in this paper were constructed using these two sources.

We will proceed in this paper by defining all the terms needed in order to state Thurston's Theorem. Then we will focus on the parts of the proof argument which pertain especially to differential topology. We will use this framework as motivation to delve into the details of complex structures and an appropriate Teichmüller space.

2 Defining the terms and stating the theorem

There are some conditions we need to add to the map $f : \mathbb{S}^2 \to \mathbb{S}^2$ so that we can properly state Thurston's theorem. The first relates to the dynamics that f induces on the sphere.

We denote the k^{th} iterate of f as

$$f^k := \underbrace{f \circ f \circ \cdots \circ f}_{k \text{ times}}$$

with $f^0 = \mathrm{Id}_{\mathbb{S}^2}$. As above, let Ω_f be the set of critical points of f. Then we can define the *post-critical set* P_f.

$$P_f := \bigcup_{k=1}^{\infty} f^k(\Omega_f)$$

We say that f is *post-critically finite* if P_f is a finite set. This means that all the critical points of f are pre-periodic (meaning that some iterate is periodic).

Definition 2.1. A *Thurston map* is a post-critically finite branched covering map $f : \mathbb{S}^2 \to \mathbb{S}^2$ of degree $d \geq 2$. The degree of f is defined in the topological sense as a map on spheres.

Definition 2.2 (Equivalence)**.** Let f and g be two Thurston maps. Then f and g are *equivalent* if there are homeomorphisms $\theta, \theta' : (\mathbb{S}^2, P_f) \to (\mathbb{S}^2, P_g)$ so that the diagram below commutes and θ is isotopic to θ' relative to P_f.

$$\begin{tikzcd}
(\mathbb{S}^2, P_f) \arrow[r, "\theta'"] \arrow[d, "f"'] & (\mathbb{S}^2, P_g) \arrow[d, "g"] \\
(\mathbb{S}^2, P_f) \arrow[r, "\theta"'] & (\mathbb{S}^2, P_g)
\end{tikzcd}$$

Definition 2.3. An *orbifold* (X, ν) is an oriented surface X together with a function $\nu : X \to \{1, 2, \ldots, \infty\}$ which assigns 1 to all but a discrete set of points.

We can visualize an orbifold as a manifold with a discrete set of cone points. The "pointiness" of the cone point is proportional to the number assigned to that point by ν. If X is a compact manifold, then any discrete set must be finite. Thus, we can define the *orbifold Euler characteristic* for a compact orbifold (X, ν) by

$$\chi(X, \nu) := \chi(X) - \sum_{x \in X} \left(1 - \frac{1}{\nu(x)}\right).$$

This definition makes sense because $\{x \in X \mid \nu(x) > 1\}$ is finite, so there are only finitely many nonzero terms in the sum. If $\nu \equiv 1$, then (X, ν) is a standard manifold and the orbifold Euler characteristic agrees with the standard Euler characteristic on X.

Definition 2.4. An orbifold (X, ν) is *hyperbolic* if $\chi(X, \nu) < 0$.

We will only be considering orbifold structures on the sphere. Recall that $\chi(\mathbb{S}^2) = 2$, so if we have an orbifold $(\mathbb{S}^2, \nu)$ where there are more than 4 points x with $\nu(x) > 1$, then we have

$$\chi(\mathbb{S}^2, \nu) = 2 - \sum_{x \in \mathbb{S}^2} \left(1 - \frac{1}{\nu(x)}\right) < 2 - 4\left(\frac{1}{2}\right) = 0.$$

And this shows that $(\mathbb{S}^2, \nu)$ is hyperbolic.

Given a Thurston map $f : \mathbb{S}^2 \to \mathbb{S}^2$, we define an orbifold O_f associated to f. The main idea behind this definition is that we want to encode the combinatorial information of the dynamics of f using the function ν. To do this, we ensure that $(\deg_y f)(\nu(y))$ always divides $\nu(f(y))$.

Proposition 2.5 (Proposition 10.1.8 in [2])**.** *Let $f : \mathbb{S}^2 \to \mathbb{S}^2$ be a Thurston map with post-critical set P_f. Then there is a smallest function $\nu_f : \mathbb{S}^2 \to \mathbb{N} \cup \{\infty\}$ such that*

- $\nu_f(x) = 1$ *if $x \neq P_f$*

- *For all $x \in \mathbb{S}^2$ and all $y \in f^{-1}(x)$, $\nu_f(x)$ is a multiple of* $(\deg_y f)(\nu_f(y))$.

Given a Thurston map f, we define $O_f := (\mathbb{S}^2, \nu_f)$ to be this orbifold where ν_f is the minimal function given by 2.5.

Consider the example given in Figure 1b. Let $f : \mathbb{S}^2 \to \mathbb{S}^2/D_3$ be the quotient map. To determine the orbifold O_f, first we need to identify the images of the critical points with points in $\mathbb{S}^2$. There are three orbits of critical points under the group action D_3; they are color-coded in Figure 2. Let Ω_a be the set of degree 3 points in green, Ω_b be the set of degree 2 points in blue, and Ω_c be the set of degree 2 points in pink. Then we declare $f(\Omega_a) = a$, $f(\Omega_b) = b$ and $f(\Omega_c) = c$. Then note that $\Omega_f = \Omega_a \cup \Omega_b \cup \Omega_c$ and $P_f = \{a, b, c\}$. We can now determine ν_f.

$$\nu_f(x) = \begin{cases} 1 & x \notin P_f \\ \infty & x = a, b, c \end{cases}$$

Then we can calculate the orbifold Euler characteristic of $O_f = (\mathbb{S}^2, \nu_v)$.

$$\chi(O_f) = 2 - 3\left(1 - \frac{1}{\infty}\right) = -1$$

And so we see that O_f is a hyperbolic orbifold.

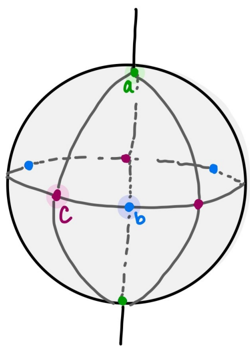


Figure 2: Identifying the images of the critical points of $f : \mathbb{S}^2 \to \mathbb{S}^2/D_3$ with points on the sphere. Each color (green, blue, pink) represents a different orbit under the action by D_3.

The last definition that we need in order to state Theorem 2.6 is for f-stable multicurves. We will give a very quick definition for now, because we will not be returning to this idea in (this iteration) of this paper.

A simple closed curve γ on $\mathbb{S}^2 \setminus P_f$ is *nonperipheral* if both connected components of its complement contain at least two critical points. A *multicurve* Γ on $(\mathbb{S}^2, P_f)$ is a collection of disjoint, nonhomotopic, nonperipheral, simple closed curves on $\mathbb{S}^2 \setminus P_f$. Such a multicurve Γ is *f-stable* if for all $\gamma \in \Gamma$, each connected component of $f^{-1}(\gamma)$ is peripheral or homotopic to a curve in Γ relative to P_f.

Given an f-stable multicurve Γ, we form a vector space $\mathbb{R}^\Gamma$ where the elements are formal linear combinations of curves in Γ. Then, because Γ is f-stable, f induces a linear transformation on this vector space. Specifically, this is given by the formula below. Here, $\Gamma_\gamma(\delta)$ is the set of connected components of $f^{-1}(\gamma)$ which are homotopic to δ relative to P_f.

$$f_\Gamma([\gamma]) := \sum_{\delta\in\Gamma}\left(\sum_{\eta\in\Gamma_\gamma(\delta)}\frac{1}{\deg(f|_\eta : \eta\to\gamma)}\right)[\delta]$$

Then $f_\Gamma : \mathbb{R}^\Gamma \to \mathbb{R}^\Gamma$ is a linear transformation, so it can be represented as a matrix once we order the curves $\gamma_1, \ldots, \gamma_n \in \Gamma$. This matrix will have nonnegative entries and so, by the Perron-Frobenius theorem, it has a nonnegative real largest eigenvalue. We call this eigenvalue λ_Γ.

There is intuition for why we define f_Γ and λ_Γ in this way, and why they are the keystone for the following theorem. They are not the main focus for this paper, but we encourage the reader to look at Hubbard's exposition in [2] if interested.

Theorem 2.6 (Topological characterization of rational functions[1])**.** *Let $f : \mathbb{S}^2 \to \mathbb{S}^2$ be a post-critically finite branched covering of the sphere with hyperbolic orbifold O_f. Then f is equivalent to a rational function if and only if for every f-stable multicurve Γ on $(\mathbb{S}^2, P_f)$, the eigenvalue λ_Γ satisfies $\lambda_\Gamma < 1$.*

If f is equivalent to a rational function g, then g is unique up to conjugation by an automorphism of the Riemann sphere $\mathbb{P}^1$.

3 Teichmüller space

The proof for Theorem 2.6 proceeds by considering the map that f induces on a certain Teichmüller space: the space of all marked complex structures on $\mathbb{S}^2$ with finitely many marked points. This induced map, σ_f, has a fixed point unless we have an f-stable multicurve Γ with eigenvalue $\lambda_\Gamma = 1$. Such a fixed point corresponds to a rational function equivalent to f. To begin understanding the proof, we define the Teichmüller space and the map σ_f.

Definition 3.1 (Teichmüller space, version 1)**.** Let $\mathcal{T}_f$ be the Teichmüller space on $\mathbb{S}^2$ with marked points P_f. That is, $\mathcal{T}_f$ is the space of diffeomorphisms $\phi : (\mathbb{S}^2, P_f) \to \mathbb{P}^1$ with ϕ_1 and ϕ_2 identified if and only if there is an analytic isomorphism $h : \mathbb{P}^1 \to \mathbb{P}^1$ so that $h \circ \phi_1|_{P_f} = \phi_2|_{P_f}$ and $h \circ \phi_1$ is isotopic to ϕ_2 relative to P_f.

$$\begin{tikzcd}
(\mathbb{S}^2, P_f) \arrow[r, "\phi_1"] \arrow[dr, "\phi_2"'] & \mathbb{P}^1 \arrow[d, "h"] \\
& \mathbb{P}^1
\end{tikzcd}$$

Remark 3.2 (Recalling the definition of isotopy)*.* Two maps ψ_1 and ψ_2 in $\mathrm{Diff}(\mathbb{S}^2 \to \mathbb{P}^1)$ are *isotopic* if there is a continuous map $\gamma : [0,1] \to \mathrm{Diff}(\mathbb{S}^2 \to \mathbb{P}^1)$ with $\gamma(0) = \psi_1$ and $\gamma(1) = \psi_2$. They are *isotopic relative to* $P \subset \mathbb{S}^2$ if $\psi_1|_P = \psi_2|_P$ and $\gamma(t)|_P = \psi_1|_P$ for all $t \in [0,1]$.

Definition 3.3. A *complex structure* μ on a manifold M with real dimension $2m$ is smooth structure where the transition functions are holomophic when viewed as maps on $\mathbb{C}^m \cong \mathbb{R}^{2m}$. Practically, μ

[1] Appears as Theorem 10.1.14 in Teichmüller TheoryVolume 2

consists of local charts (U, φ) where $\varphi : U \to \mathbb{C}^m$ is a homeomorphism onto an open set of $\mathbb{C}^m$ and for two local charts $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β), the transition map

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \mathbb{C}^m$$

is holomorphic.

Definition 3.4 (Teichmüller sapce, version 2)**.** We can also define $\mathcal{T}_f$ as the space of complex structures on $\mathbb{S}^2$, up to an equivalence relation. Let μ_1 and μ_2 be two complex structures on $\mathbb{S}^2$. Then μ_1 and μ_2 are identified if $\mu_1 = h^*\mu_2$ for a diffeomorphism $h : \mathbb{S}^2 \to \mathbb{S}^2$ with $h|_{P_f} = \mathrm{id}$ and h isotopic to the identity relative to P_f.

Theorem 3.5 (Uniformization Theorem)**.** *Every simply connected surface with a complex structure is conformally equivalent to one of the following: the unit disk $\mathbb{D}$, the complex plane $\mathbb{C}$, or the Riemann sphere $\mathbb{P}^1$.*

Remark 3.6. We want to emphasize that in both definitions above, we have isotopy *relative to P_f*. We imagine the points of P_f as being "pinned down" as we deform the map h. If we were to "lift up" the pins, then h will always be isotopic to the identity. This follows from the Uniformization Theorem 3.5: any topological sphere with a complex structure is conformally equivalent to $\mathbb{P}^1$. Hence, there is a unique complex structure on $\mathbb{S}^2$ without any marked points, so the Teichmüller space consists of a single point.

We can see the equivalence between these two definitions of the Teichmüller space by pulling back the complex structure on $\mathbb{P}^1$.

Proposition 3.7. *The two definitions of $\mathcal{T}_f$ discussed above— Definition 3.1 and Definition 3.4— are equivalent descriptions.*

Proof. Let $\mu_0 = \{(U_\alpha, \varphi_\alpha)\}$ the unique complex structure on $\mathbb{P}^1$ as discussed in Remark 3.6. We construct a correspondence map $\{\phi : \mathbb{S}^2 \to \mathbb{P}^1 \text{ is a diffeomorphism}\} \leftrightarrow \{\mu \text{ complex structure on } \mathbb{S}^2\}$ which factors through the quotients nicely.

($\to$) Given $\phi : \mathbb{S}^2 \to \mathbb{P}^1$, we define a complex structure on $\mathbb{S}^2$. Let

$$\mu := \phi^*\mu_0 \quad \text{be the collection of local charts} \quad \{(\phi^{-1}(U_\alpha), \varphi_\alpha \circ \phi)\}.$$

Note that each $\phi^{-1}(U_\alpha)$ is an open set in $\mathbb{S}^2$ homeomorphic to a disk because ϕ is a diffeomorphism. Also, the transition maps are given by $(\varphi_\beta \circ \phi) \circ (\varphi_\alpha \circ \phi)^{-1} = \varphi_\beta \circ \varphi_\alpha^{-1}$ so these are holomorphic. Thus, μ is a well-defined complex structure on $\mathbb{S}^2$.

Then suppose that ϕ_1, ϕ_2 are two representatives for the same point in $\mathcal{T}_f$. We want to show they pullback the same complex structure on $\mathbb{S}^2$, up to the appropriate equivalence relation. Let $h : \mathbb{P}^1 \to \mathbb{P}^1$ be an analytic isomorphism satisfying the conditions in Definition 3.1. Then we define $\tilde{h} := \phi_2^{-1} \circ h \circ \phi_1$.

$$\begin{array}{ccc} \mathbb{S}^2 & \xrightarrow{\phi_1} & \mathbb{P}^1 \\ {\scriptstyle\tilde{h}}\downarrow & & \downarrow{\scriptstyle h} \\ \mathbb{S}^2 & \xrightarrow[\phi_2]{} & \mathbb{P}^1 \end{array}$$

We are given that $\phi_2|_{P_f} = h \circ \phi_1|_{P_f}$ so we have $\tilde{h}|_{P_f} = (\phi_2^{-1} \circ h \circ \phi_1)|_{P_f} = \mathrm{id}_{P_f}$. And similarly, ϕ_2 is isotopic to $h \circ \phi_1$ relative to P_f so $\tilde{h} = \phi_2^{-1} \circ h \circ \phi_1$ is isotopic to the identity relative to P_f.

($\leftarrow$) Next, suppose we are given a complex structure μ on $\mathbb{S}^2$. Then $\mathbb{S}^2$ with this structure is a complex manifold which is homeomorphic to $\mathbb{P}^1$. So by the Uniformization Theorem (3.5), it is conformally equivalent to $\mathbb{P}^1$. So we can choose a conformal diffeomorphism $\phi : \mathbb{S}^2_\mu \to \mathbb{P}^1$.

Let μ_1 and μ_2 be two equivalent complex structures on $(\mathbb{S}^2, P_f)$ and let $h : \mathbb{S}^2 \to \mathbb{S}^2$ be a diffeomorphism realizing this equivalence. Since $h^*\mu_2 = \mu_1$, this is also an analytic isomorphism when the complex structures are taken into account. Let $\phi_1, \phi_2 : \mathbb{S}^2 \to \mathbb{P}^1$ be the maps chosen for μ_1 and μ_2. Then we define $\tilde{h} := \phi_2 \circ h \circ \phi_1^{-1}$. We can then check that $\tilde{h}$ satisfies the conditions in Definition 3.1. First, note that

$$\tilde{h} \circ \phi_1 = \phi_2 \circ h.$$

Then, because $h|_{P_f} = \mathrm{id}$ and h is isotopic to the identity relative to P_f, we have $\phi_2 \circ h|_{P_f} = \phi_2|_{P_f}$ and $\phi_2 \circ h$ is isotopic to ϕ_2 relative to P_f. $\square$

Proposition 3.8 (Prop 2.1 in paper)**.** *For a Thurston map $f : \mathbb{S}^2 \to \mathbb{S}^2$, and a complex structure $\mu \in \mathcal{T}_f$, there is a well-defined pullback of the complex structure $f^*\mu \in \mathcal{T}_f$. This pullback induces an analytic mapping $\sigma_f : \mathcal{T}_f \to \mathcal{T}_f$.*

Proof. We will define $f^*\mu$ on open sets, and then we check that the transition functions are smooth. First, suppose that $U \subset \mathbb{S}^2$ is an open ball which contains no critical points ($U \cap \Omega_f = \emptyset$). Then $f|_U$ is a diffeomorphism and so we can pullback the complex structure μ onto U by f. That is, if $(U_\alpha, \varphi_\alpha)$ is a local chart on $f(U)$, then

$$(V_\alpha, \varphi_\alpha \circ f) \quad \text{where} \quad V_\alpha = U \cap f^{-1}(U_\alpha)$$

is a local chart on U.

Next, suppose that $\omega \in \Omega_f$ is a critical point and that U is a neighborhood of ω such that $U \setminus \{\omega\} \cap \Omega_f = \emptyset$. Then we need to define a local chart on a small ball containing ω.

Let (W, φ) be a local chart in $f(U)$ with $f(\omega) \in W$. Note that $f^{-1}(W)$ is an open ball in U containing ω. By the Uniformization Theorem, we can assume that $\varphi : W \to \mathbb{D}$ is a conformal diffeomorphism and $\varphi(f(\omega)) = 0$. Then, letting $\deg_\omega f = k$, we can choose a homeomorphism $\varphi' : f^{-1}(W) \to \mathbb{D}$ with $\varphi'(\omega) = 0$ such that the diagram below commutes. We can do this because we know how to choose a branch of the inverse map $w \mapsto w^{1/k}$.

$$\begin{tikzcd}
f^{-1}(W) \arrow[r, "f"] \arrow[d, "\varphi'"'] & W \arrow[d, "\varphi"] \\
\mathbb{D} \arrow[r, "z \mapsto z^k"'] & \mathbb{D}
\end{tikzcd}$$

Then $(f^{-1}(W), \varphi')$ gives local coordinates at ω in the complex structure $f^*\mu$. We claim that this construction is independent of the choice of representative for μ, and is thus well-defined. Suppose that ϕ_1 and ϕ_2 two representatives for the same point in $\mathcal{T}_f$. Let $h : \mathbb{P}^1 \to \mathbb{P}^1$ be a map realizing this equivalence. Then $(h \circ \phi_1 \circ f)|_{P_f} = (\phi_2 \circ f)|_{P_f}$ because $(h \circ \phi_1)|_{P_f} = \phi_2|_{P_f}$ and $f(P_f) \subset P_f$. And, in a similar fashion, we have that $h \circ \phi_1 \circ f$ is isotopic to $\phi_2 \circ f$ relative to P_f.

$$\begin{tikzcd}
\mathbb{S}^2 \arrow[r, "f"] & \mathbb{S}^2 \arrow[r, "\phi_1"] \arrow[dr, "\phi_2"'] & \mathbb{P}^1 \arrow[d, "h"] \\
 & & \mathbb{P}^1
\end{tikzcd}$$

Then we define $\sigma_f : \mathcal{T}_f \to \mathcal{T}_f$ to be this pullback map. If $\tau \in \mathcal{T}_f$ is represented by the complex structure μ, then $\sigma_f(\tau)$ is represented by the complex structure $f^*\mu$. $\square$

Given a point $\tau \in \mathcal{T}_f$, let ϕ be a diffeomorphism representing τ. Then let ϕ' be a diffeomorphism representing $\sigma_f(\tau)$. We will sometimes write that $\sigma_f(\phi) = \phi'$. Then we define $f_\tau := \phi \circ f \circ (\phi')^{-1}$ and we draw the following commutative diagram.

$$\begin{array}{ccc} (\mathbb{S}^2, P_f) & \xrightarrow{\phi'} & \mathbb{P}^1 \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f_\tau} \\ (\mathbb{S}^2, P_f) & \xrightarrow{\phi} & \mathbb{P}^1 \end{array}$$

This diagram encodes the action of σ_f. Also, f_τ is a rational map of degree $d = \deg f$. This follows because it is meromorphic on $\mathbb{P}^1$ by construction. These observations are the heart of the proof for Proposition 3.10.

Proposition 3.9. *If f and g are equivalent and θ, θ' realize this equivalence (as in Definition 2.2), then*

$$\theta^* = \theta'^* : \mathcal{T}_g \to \mathcal{T}_f$$

is an isomorphism such that $\theta^* \circ \sigma_g = \sigma_f \circ \theta^*$.

Proposition 3.10 (Prop 2.3 in the paper)**.** *A Thurston map f is equivalent to a rational function if and only if σ_f has a fixed point.*

Proof. ($\Leftarrow$) Suppose first that f is equivalent to a rational map g. Let θ and θ' be diffeomorphisms

$$\theta, \theta' : (\mathbb{S}^2, P_f) \to (\mathbb{P}^1, P_g)$$

that realize this equivalence. Consider the diagram below.

$$\begin{array}{ccc} (\mathbb{S}^2, P_f) & \xrightarrow{\theta'} & (\mathbb{P}^1, P_g) \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ (\mathbb{S}^2, P_f) & \xrightarrow[\theta]{} & (\mathbb{P}^1, P_g) \end{array}$$

Notice that because the diagram commutes and g is meromorphic with respect to the standard complex structure on $\mathbb{P}^1$, we have $\sigma_f(\theta) = \theta'$. But also, by the definition of equivalence (2.2), θ is isotopic to θ' relative to P_g. So θ is a fixed point for σ_f.

($\Rightarrow$) Suppose now that σ_f has a fixed point in $\mathcal{T}_f$ and let this be represented by $\phi : (\mathbb{S}^2, P_f) \to \mathbb{P}^1$. Then define $\phi' = \sigma_f(\phi)$ and ϕ' is identified with ϕ in $\mathcal{T}_f$ since ϕ is a fixed point of σ_f. Let h realize this equivalence. Consider the following commutative diagram.

$$\begin{tikzcd} & \mathbb{P}^1 \arrow[d, "h"] \\ (\mathbb{S}^2, P_f) \arrow[ur, "\phi"] \arrow[r, "\phi'"] \arrow[d, "f"'] & \mathbb{P}^1 \arrow[d, "f_\tau"] \\ (\mathbb{S}^2, P_f) \arrow[r, "\phi"] & \mathbb{P}^1 \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} (\mathbb{S}^2, P_f) \arrow[r, "h^{-1}\circ\phi'"] \arrow[d, "f"'] & \mathbb{P}^1 \arrow[d, "f_\tau \circ h"] \\ (\mathbb{S}^2, P_f) \arrow[r, "\phi"] & \mathbb{P}^1 \end{tikzcd}$$

Note that $h^{-1} \circ \phi'$ and ϕ are diffeomorphisms. Also, $f_\tau \circ h$ is a rational map because it is meromorphic. Finally, $h^{-1} \circ \phi'$ and ϕ are isotopic relative to P_f by construction. Thus, f is equivalent to the rational map $f_\tau \circ h$. $\square$

References

[1] Adrien Douady and John H. Hubbard. A proof of Thurston's topological characterization of rational functions. *Acta Mathematica*, 171(2):263 – 297, 1993.

[2] John H. Hubbard. *Teichmüller Theory Volume 2: Surface Homeomorphisms and Rational Functions*. Matrix Editions, 2016.