All cellular strings on polytopes at once

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December 11, 2022

Abstract

For a linear functional f on a polytope P, a cellular string is a list of faces of P from minimum to maximum, according to f. We can order all f-cellular strings by inclusion and we call this the Baues poset $\omega(P, f)$. In [BKS94] the authors proved that for an n-dimensional polytope, $\omega(P, f)$ is homotopy equivalent to $\mathbb{S}^{n-2}$. We expand this result by considering all cellular strings on P for any linear functional; we call this set the mega Baues poset $\Omega(P)$. We claim that this space is homotopy equivalent to the unit tangent bundle on $\mathbb{S}^{n-1}$. This paper contain the bulk of the proof and the details will be completed in a later paper.

1 Initial Definitions

Definition 1.1. A *convex polytope* is the convex hull of a finite number of points in $\mathbb{R}^n$. More specifically, if we are given a point set $\{v_1, \ldots, v_k\} \subset \mathbb{R}^n$, then its convex hull is given by

$$P := \left\{ \lambda_1 v_1 + \cdots + \lambda_k v_k \;\middle|\; \lambda_i \geq 0,\ \sum_{i=1}^{k} \lambda_i = 1 \right\}$$

and this is a convex polytope.

Although this definition seems daunting, polytopes are likely familiar for the reader. The triangle, square, and pentagon are 2-dimensional polytopes. The tetrahedron and cube are 3-dimensional polytopes (see Figure 1). In two dimensions, the convex hull of a finite point set can be thought of as stretching a rubber band around some fixed points and letting it go (see Figure 2). In three dimensions we can think of stretching a latex balloon around a finite number of fixed points in space. For a concrete example, take the following point set in $\mathbb{R}^3$.

$$\Big\{(0,0,0),\ (1,0,0),\ (0,1,0),\ (0,0,1),\ (1,1,0),\ (1,0,1),\ (0,1,1),\ (1,1,1)\Big\}.$$

The convex hull of these points is the standard unit cube with these eight points becoming the vertex set.

A *face* of a polytope P is the convex hull of a subset of its vertices which does not intersect the interior of P. The dimension of a face is the dimension of the smallest affine subspace of $\mathbb{R}^n$ containing the face. For a face of dimension k, we will refer to it as a k-face. For example, the vertex $(0,0,0)$ is a 0-face of the cube; the edge from $(0,0,0)$ to $(1,0,0)$ is a 1-face of the cube; and the unit square with vertices $\{(0,0,0),\ (1,0,0),\ (0,1,0),\ (1,1,0)\}$ is a 2-face of the cube.

Now we turn our attention to the other main piece of this investigation.

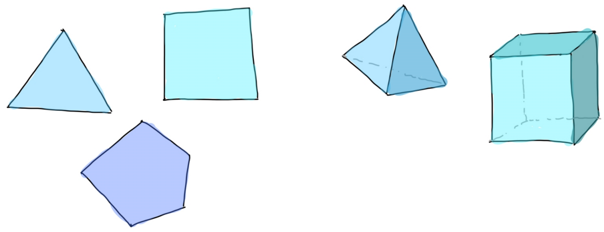


Figure 1: Two-dimensional polytopes (triangle, square, pentagon) on the left. Three-dimensional polytopes (tetrahedron, cube) on the right.

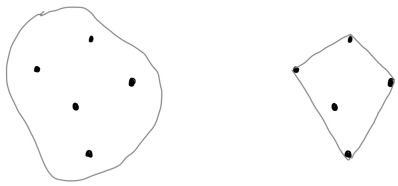


Figure 2: An intuitive picture of taking the *convex hull* in two dimensions.

Definition 1.2. A *linear functional* is a linear map $f : \mathbb{R}^n \to \mathbb{R}$. A map f is linear if for all $x_1, x_2 \in \mathbb{R}^n$ and $a \in \mathbb{R}$, we have

$$f(x_1 + x_2) = f(x_1) + f(x_2) \quad \text{and} \quad f(ax_1) = af(x_1).$$

For any linear functional $f : \mathbb{R}^n \to \mathbb{R}$, there is a unique vector $a \in \mathbb{R}^n$ such that $f(x) = a \cdot x$ for all $x \in \mathbb{R}^n$. The vector $a = (a_1, \ldots, a_n)$ is defined such that $a_i = f(e_i)$ where e_i is the standard basis vector in $\mathbb{R}^n$. We leave it to the reader to show that a has the properties we claim.

In this paper, we will apply a linear functional $f : \mathbb{R}^n \to \mathbb{R}$ to an n-dimensional polytope P in $\mathbb{R}^n$. We then examine which faces of P are maximized or minimized under f. Recall that the dot product of two unit vectors is maximized when they point in the same direction and the dot product is minimized when they point in opposite directions. Consider Figure 3a: which vertex of P is maximized under the linear functional associated to vector a? Which vertex is minimized?

Given a polytope P and a vector a, we can visually determine how the vertices on P are ordered according to the linear functional. First, extend vector a to a line. Next, project each vertex of P orthogonally onto this line. The direction of a indicates the maximum end of the line. Note that this process is a direct interpretation of the physical meaning of the dot product: $a \cdot x$ is the "component of x in the direction of a". In this way, applying a linear functional to a polytope can be thought of as "squishing" the polytope onto a line segment. Figure 3b shows the result of performing this process on the example in Figure 3a. With this construction, we see that v_2 is the maximum vertex and v_4 is the minimum vertex.

For a fixed polytope P in $\mathbb{R}^n$, a linear functional $f : \mathbb{R}^n \to \mathbb{R}$ is *generic* if f is non-constant on every k-face of P for $k \geq 1$. For the first part of this paper, we will only discuss generic linear

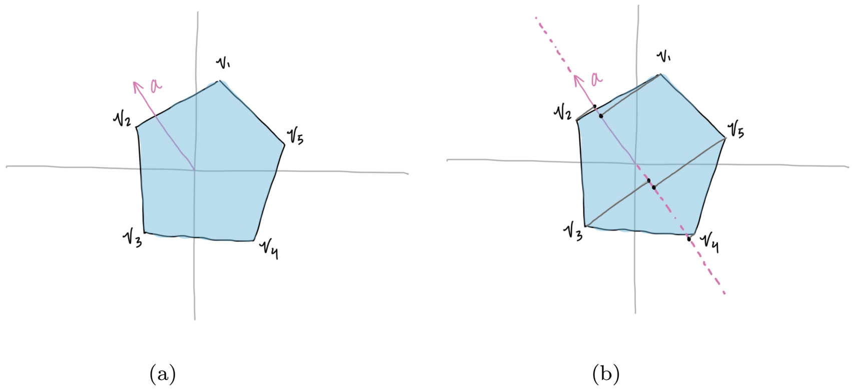


Figure 3: We can find the maximum and minimum vertices of a polytope P according to a linear functional a by projecting each vertex orthogonally onto the line containing a.

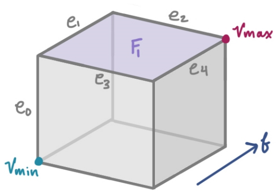


Figure 4: The monotone paths $\{e_0, e_1, e_2\}$ and $\{e_0, e_3, e_4\}$ are related by a *face flip* across F_1.

functionals on polytopes. A generic linear functional achieves its maximum (resp. minimum) at a unique vertex on the polytope. This means that we can more easily define the following objects.

Definition 1.3. Let P be a polytope and f be a generic linear functional on P. Then a path comprised of edges $\{e_1, \ldots, e_k\}$ is a *monotone path* if $v_{\min} \in e_1$, $v_{\max} \in e_k$, and f is strictly increasing over each e_j.

Sometimes it helps to think about f as describing a temperature gradient on the polytope P. The coldest point on P is $v_{\min}$ and the hottest point is $v_{\max}$. Then a monotone path is a walk along the edges of P from the coldest point to the hottest where the temperature is always increasing.

Figure 4 shows a linear functional on a cube with minimum and maximum vertices as labeled. Consider two monotone paths on this cube given by $P_1 = \{e_0, e_1, e_2\}$ and $P_2 = \{e_0, e_3, e_4\}$. The difference between these two paths depends on whether we walk around F_1 in the clockwise or counterclockwise direction. We say that P_1 and P_2 are related by a *face flip* because we can imagine reflecting F_1 over a diagonal axis to turn P_1 into P_2. What if we walked from $v_{\min}$ to $v_{\max}$ using the edge e_0 and the entire face F_1? This walk would contain both P_1 and P_2. From this line of reasoning, we arrive at the following definition.

Definition 1.4. A *cellular string* on P is a sequence of faces $F_1, \ldots, F_k$ such that

(i) $v_{\min} \in F_1$ and $v_{\max} \in F_k$;

(ii) f is non-constant on all F_i;

(iii) the f-maximizing face of F_i is the f-minimizing face of F_{i+1} for $1 \leq i \leq k-1$.

Remark. *This last condition guarantees that the cellular string will be connected and that it will not "back track". We did not need this condition when defining monotone paths because there is only one direction to "travel" along an edge.*

If $\mathcal{F}$ is a cellular string on polytope P according to a linear functional f, then we will sometimes call $\mathcal{F}$ an f-cellular string. Looking back at the example in Figure 4, the list $\{e_0, F_1\}$ is a cellular string. The two monotone paths P_1 and P_2 are "children" of this cellular string. We make this notion more formal in the next section.

In light of definition 1.4, we see that the actual value of f at each vertex of P is not relevant to the creation of cellular strings: only the ordering, from minimum to maximum, that f imposes on the vertices of P. Since we are only concerned with the set of cellular strings on a polytope, moving forward we will not find the specifics of the linear functional relevant. In particular, for $c > 0$, $c \cdot f$ and f will designate the same order on the vertices. Hence, the *magnitude* of the vector a will have no effect on the set of cellular strings– but its *direction* will be of primary importance.

2 The Baues Problem

A *partially ordered set*, commonly called a poset, is a set X along with a partial order[1] denoted by $\leq$. The partial order must satisfy three conditions. For $x, y, z \in X$, the partial order must be

- reflexive: $x \leq x$;
- antisymmetric: if $x \leq y$ and $y \leq x$, then $x = y$;
- and transitive: if $x \leq y$ and $y \leq z$ then $x \leq z$.

We denote a poset by $(X, \leq)$. A *chain* is a subset C of X where all the elements in C are comparable by the partial order. Posets are commonly used in combinatorics to organize a set into an object with more structure. A standard example is the *face poset of a polytope* P: all the faces of P ordered by inclusion. For two faces F and F' of P, we say $F \leq F'$ if and only if $F \subseteq F'$. The face poset of a polytope carries all of the combinatorial information about the polytope and simplifies many arguments about polytopes. We will use a poset to organize the set of all cellular strngs.

For two f-cellular strings $\mathcal{F} = (F_1, \ldots, F_k)$ and $\mathcal{E} = (E_1, \ldots, E_m)$, we say $\mathcal{F} \leq \mathcal{E}$ if and only if $\bigcup F_i \subseteq \bigcup E_i$. Since the polytope P is finite, there are only finitely many f-cellular strings. Letting $\{\mathcal{F}\}$ be the collection of all f-cellular strings on P, we construct the poset $\omega(P, f) := (\{\mathcal{F}\}, \leq)$ where $\leq$ is the partial order defined above. This construction is called the *Baues Poset* after H. J. Baues who studied it in his 1980 paper "Geometry of loop spaces and the cobar construction" [Bau80].

There is a standard way of constructing a simplicial complex from a poset $(X, \leq)$ called the *order complex*. The vertices of the order complex are elements of X and the higher dimensional faces are the chains in $(X, \leq)$. See Figure 5 for an example of the order complex.

Example 2.1. Let P be a tetrahedron with vertices labeled 1, 2, 3, and 4 and let f be a generic linear functional such that $f(1) < f(2) < f(3) < f(4)$. Then the cellular strings are

$$\{12, 23, 34\},\ \{12, 24\},\ \{13, 34\},\ \{14\},\ \{12, 234\},\ \{123, 34\},\ \{124\},\ \{134\}.$$

The poset $\omega(P, f)$ and its order complex are shown in Figure 6.

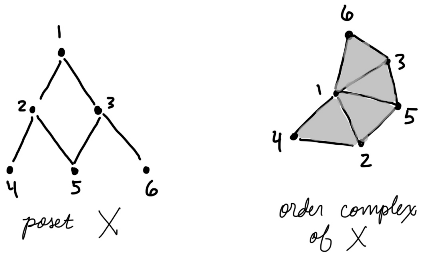


Figure 5: A poset X (left) and its order complex (right).

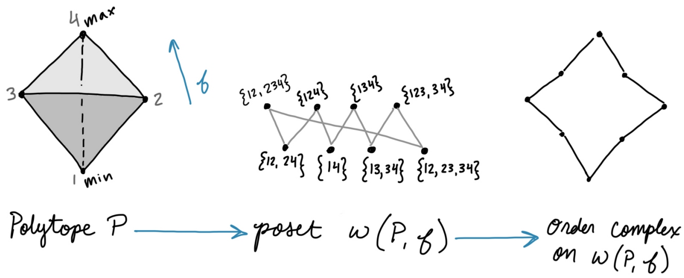


Figure 6: Constructing the Baues poset on the simplex for a generic linear functional.

The order complex allows us to view a poset as a topological object. We define the homotopy type of a poset to be the homotopy type of the order complex on that poset. In [Bau80], Baues conjectured that $\omega(P, f)$ is homotopy equivalent to a sphere. L. J. Billera, M. M. Kapranov, and B. Sturmfels gave a proof in the affirmative for this conjecture in their 1994 paper "Cellular Strings on Polytopes" [BKS94].

Theorem 2.2. *[Theorem 1.2 in [BKS94]] Let P be an n-dimensional convex polytope $P \subseteq \mathbb{R}^n$ and let $f : \mathbb{R}^n \to \mathbb{R}$ be a generic linear functional. Then the Baues poset $\omega(P, f)$ is homotopy equivalent to $\mathbb{S}^{n-2}$.*

We will not give a proof of this theorem, but we point to all the examples we have looked at to see it in action. Notice that the tetrahedron in Example 2.1 is a 3-dimensional polytope and its Baues poset, shown in Figure 6, is homotopy equivalent to $\mathbb{S}^1$.

Consider the Baues poset for any polygon (two-dimensional polytope) P and generic linear functional f. There are always exactly two f-cellular strings on P: one from the minimum to maximum in the clockwise direction, and one in the counterclockwise direction. These cellular strings are not comparable because they partition the edges of P. Also $\omega(P, f)$ has no chains larger than a single element, so it is isomorphic to its order complex. Thus, we see that $\omega(P, f)$ is two isolated vertices which is isomorphic to $\mathbb{S}^0$.

[1] The order is partial because not all pairs of elements in the set are comparable.

3 Extending the Baues Problem

We now look beyond the traditional Baues problem by asking if cellular strings from different linear functionals can exist in the same "world". Our goal is to construct one large structure– a *mega Baues poset*– which includes all cellular strings on P from any linear functional. Then what can we say about the homotopy type of this structure? Before we dive into the construction process, we first need to gather our materials.

3.1 Many linear functionals

To begin, let's take a closer look at all possible linear functionals on a polytope P. Each linear functional is associated to a unique vector in $\mathbb{R}^n$: the set of all these vectors V is called the *dual space.*

Let F be any k-face of P with $k \geq 1$. A linear functional is constant on F if and only if the associated vector is orthogonal[2] to F. We look at the set of all vectors in V which are orthogonal to F and this creates a $n - \dim(F)$ subspace of V. This is called the *orthogonal complement of* F and we will denote it $F^{\perp}$.

We can also consider the set of vectors in V where the associated linear functional is maximized[3] on F. We call this set the *normal cone* and denote it $C(F)$. The collection of all normal cones of faces of P is the *normal fan* of P given by

$$N(P) := \{C(F)\}_{F \in \text{faces}(P)}.$$

In general, a *polyhedral fan* $\mathcal{K}$ is a collection of convex polyhedral cones in $\mathbb{R}^n$ such that:

- If $\sigma \in \mathcal{K}$ then any face of σ is also in $\mathcal{K}$.
- If $\sigma, \tau \in \mathcal{K}$ then $\sigma \cap \tau$ is also in $\mathcal{K}$.

The normal fan $N(P)$ covers the dual space V and has reverse inclusion properties: for faces F and G of P we have $F \subseteq G$ if and only if $C(G) \subseteq C(F)$. We leave it to the reader to check that this property and the definition of normal cones imply that $N(P)$ satisfies the conditions for a polyhedral fan.

A vector $a \in C(F)$ is maximized on F if and only if $-a$ is minimized on F. Hence we define

$$-C(F) := \{-a \mid a \in C(F)\}$$

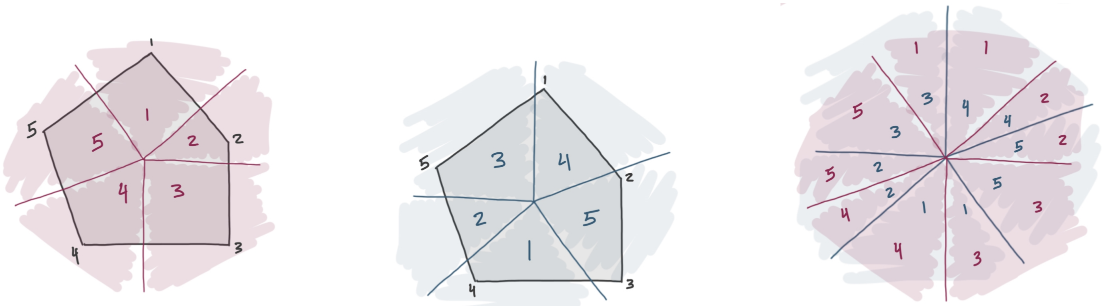
to be the set of all vectors minimized on face F. The collection $-N(P) = \{-C(F)\}$ is also a normal fan since $N(P)$ is. Once we have $N(P)$ and $-N(P)$, we can use them to construct another polyhedral fan $\Gamma(P)$ which includes information from both of them.

$$\Gamma(P) := \{C(F) \cap -C(G) \mid F, G \text{ are faces of } P\}$$

This fan $\Gamma(P)$ is called the *common refinement* of $N(P)$ and $-N(P)$. A face of $\Gamma(P)$, given by $C(F) \cap -C(G)$, contains exactly the set of linear functionals with maximum at F and minimum at G. Hence, $\Gamma(P)$ partitions all linear functionals in V based on their minimum and maximum faces. Note that if a achieves its maximum (or minimum) on a face F, then a is also constant on F. This means that geometrically, $\Gamma(P)$ is made from convex polyhedral cones orthogonal to each face of P.

[2] Exercise for the reader to prove this!

[3] This means that $f(x) \geq f(y)$ for all $x \in F$ and $y \in P$. By the convexity of P, f will be constant on F.



(a) The normal fan $N(P)$ overlayed on polytope P.

(b) $-N(P)$ overlayed on P.

(c) $\Gamma(P)$

Figure 7: Creating $\Gamma(P)$ for a polygon P.

Example 3.1 (*Creating* $\Gamma(P)$ *for a polygon* P)**.** First we create $N(P)$. Since P is a polygon, $C(e)$ for an edge e will be a ray orthogonal to e in the direction of e. We start by drawing each of these rays issuing from the origin in V. These subdivide V into one region for each vertex of P. For a vertex v of P contained in edges e_1 and e_2, $C(v)$ will be the region bounded by $C(e_1)$ and $C(e_2)$. Figure 7a shows this construction for a pentagon P with $N(P)$ overlayed on top of P. The red numbers indicate the vertex which is maximized in that region.

Next, we create $-N(P)$. In two-dimensions, $-N(P)$ will be a 180° rotation of $N(P)$. Figure 7b shows $-N(P)$ overlayed on P. The blue numbers indicate the vertex which is minimized in that region of $-N(P)$.

Finally, we take the common refinement of $N(P)$ and $-N(P)$ to get $\Gamma(P)$. This can be visualized by overlaying the two polyhedral fans and looking at the intersections of their faces. Figure 7c depicts the result with the red numbers indicating the vertex which is maximized in that region and the blue numbers indicating the vertex which is minimized.

We have now "grouped together" linear functionals in V which have similar properties. However, there can be two linear functionals in the same face of $\Gamma(P)$—both with the same minimum and maximum faces on P—which generate different sets of cellular strings on P. Take, for example, linear functionals f_1 and f_2 on the tetrahedron in Figure 8. The vertices are ordered differently by f_1 and f_2:

$$f_1(1) < f_1(3) < f_1(2) < f_1(4) \quad \text{and} \quad f_2(1) < f_2(2) < f_2(3) < f_2(4).$$

Although, f_1 and f_2 do have the same maximum and minimum vertices of P so they will be in the same face of $\Gamma(P)$. Now see that $\{1,3\},\{3,2\},\{2,4\}$ is an f_1-cellular string but not an f_2-cellular string. On the other hand, $\{1,4\}$ is both an f_1- and an f_2-cellular string. This means that we must further refine $\Gamma(P)$ if we would like linear functionals in the same face of the fan to generate the same set of cellular strings.

For a vertex v of P, every linear functional will be constant on v and so $v^{\perp} = V$. Also, for an edge e of P, the space $e^{\perp}$ will be a hyperplane in V (i.e. $n-1$ dimensional subspace). If F is a k-dimensional face with $k \geq 2$, then a linear functional is constant on F if and only if it is constant on all edges bounding F. This follows because a vector orthogonal to a set of coplanar lines is also orthogonal to the plane containing those lines. This observation implies that if F is bounded by edges $e_1, \ldots, e_m$, then $F^{\perp} = e_1^{\perp} \cap \cdots \cap e_m^{\perp}$.

Hence, for any face $F \subseteq P$, we can express $F^{\perp}$ as the intersection of hyperplanes. This means that we do not need to store information on $F^{\perp}$ for *every* face of P, but rather, we only need $e^{\perp}$

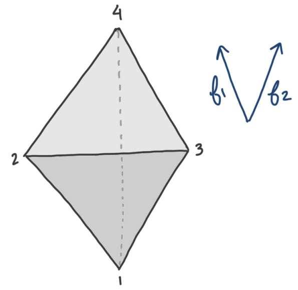


Figure 8: Two linear functionals with the same maximum and minimum faces on a tetrahedron.

for each edge. The collection $\{e^\perp\}_{e \in \mathrm{edges}(P)}$ is called a *hyperplane arrangement*. This set subdivides the space V into a polyhedral fan; we call this fan $\Sigma(P)$. Let $T_1, \ldots, T_m$ be the set of distinct hyperplanes such that $T_i = e^\perp$ for some edge e of P.[4] For each T_i, notice that $V \setminus T_i$ consists of two open half spaces. Label the closures of these regions (closed half-spaces) T_i^{-1} and T_i^{+1} arbitrarily and let $T_i^0 = T_i$. Then a face σ of $\Sigma(P)$ is defined by a list $(x_1, \ldots, x_m)$ such that $x_i \in \{-1, 0, +1\}$. We set

$$\sigma = \bigcap_{i=1}^{m} T_i^{x_i}.$$

The n-dimensional faces of $\Sigma(P)$ arise when $x_i \neq 0$ for all i. Although, depending on the geometry of the hyperplanes, we can get $\sigma = \{0\}$ in this case. Similarly, if there are exactly k entries x_i for which $x_i = 0$, then $\dim(\sigma) \leq k$.

Example 3.2 (*Creating* $\Sigma(P)$ *for a polygon* P)**.** Consider the polygon P with edge labels as shown in Figure 9a. Since no two edges of P are parallel, each edge with have a distinct orthogonal complement. Figure 9b shows $T_1 := e_1^\perp$ and the corresponding regions T_1^{-1} and T_1^{+1} of V. Note that the choice of $-1/+1$ was made arbitrarily and the other labeling system would also work. Then, once we have done this for all edges $e_1, \ldots, e_5$, we can construct $\Sigma(P)$ using the process described above. This is shown in Figure 9c. Notice that $\Sigma(P)$ is actually the common refinement of polyhedral fans $\{T_i^{-1}, T_i^0, T_i^{+1}\}$.

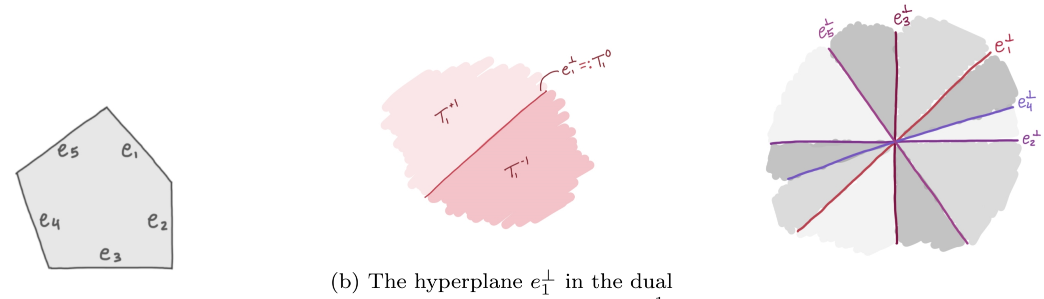
We claim that $\Sigma(P)$ is a polyhedral fan and that it is the refinement of $\Gamma(P)$ that we are looking for. That is, $\Sigma(P)$ functions as a "map"[5] for all possible linear functionals with distinct Baues posets on P. To start, we show $\Sigma(P)$ is a polyhedral fan.

Lemma 3.3. $\Sigma(P)$ *is a polyhedral fan.*

Proof. First, we show that every face of $\Sigma(P)$ is a convex polyhedral cone. Let σ be a face of $\Sigma(P)$ such that $(x_1, \ldots, x_m)$ is the list defining σ and

$$\sigma = \bigcap_{i=1}^{m} T_i^{x_i}.$$

[4] Note that some edges in P may have the same orthogonal complement so there may be fewer T_i than edges of P.
[5] in the atlas sense



(a) The polygon P with edges labeled.

(b) The hyperplane $e_1^\perp$ in the dual space V with regions labeled T_1^{-1}, T_1^0, and T_1^{+1}.

(c) $\Sigma(P)$. Hyperplanes for different edges are color-coded.

Figure 9: Creating $\Sigma(P)$ for a polygon P.

If $x_i \neq 0$, then $T_i^{x_i}$ is a closed half space. Otherwise if $x_i = 0$ then $T_i^{x_i} = T_i = T_i^{-1} \cap T_i^{+1}$ and this is the intersection of closed half spaces. Hence, we see that σ is the intersection of closed half spaces so it is a convex polyhedral cone.

Next, suppose that τ is a face of σ. If $\tau \neq \sigma$, then τ is contained in $\partial\sigma$ and

$$\tau = \left(T_{i_1} \cap \cdots \cap T_{i_\ell}\right) \cap \sigma$$

for some set of hyperplanes $T_{i_1}, \ldots, T_{i_\ell}$ bounding σ. In this case, we have

$$\tau = \bigcap_{i=1}^{m} T_i^{y_i} \quad \text{such that } y_i = \begin{cases} 0 & \text{if } i = i_j \text{ for } 1 \leq j \leq \ell \\ x_i & \text{if not} \end{cases}.$$

Then τ is the face of $\Sigma(P)$ defined by the list $(y_1, \ldots, y_m)$.

Lastly, suppose that σ and τ are faces of $\Sigma(P)$ defined by lists $(x_1, \ldots, x_m)$ and $(y_1, \ldots, y_m)$ respectively. Then

$$\sigma = \bigcap_{i=1}^{m} T_i^{x_i} \quad \text{and} \quad \tau = \bigcap_{i=1}^{m} T_i^{y_i}$$

so we have

$$\sigma \cap \tau = \bigcap_{i=1}^{m} T_i^{x_i} \cap T_i^{y_i}.$$

Notice that if $x_i = y_i$, then $T_i^{x_i} \cap T_i^{y_i} = T_i^{x_i}$. Otherwise, if $x_i \neq y_i$, then $T_i^{x_i} \cap T_i^{y_i} = T_i = T_i^0$. Hence, we see that $\sigma \cap \tau = \omega$ where ω is the face of $\Sigma(P)$ defined by the list

$$(z_1, \ldots, z_m) \quad \text{such that } z_i = \begin{cases} x_i & \text{if } x_i = y_i \\ 0 & \text{if } x_i \neq y_i \end{cases}.$$

Therefore, we have seen that $\Sigma(P)$ satisfies all the conditions and it is a polyhedral fan. □

Lemma 3.4. *$\Sigma(P)$ is a refinement of $\Gamma(P)$. That is, for all $\sigma \in \Sigma(P)$ and $\tau \in \Gamma(P)$, $\sigma \cap \tau$ is a face of $\Sigma(P)$.*

Proof. Let σ be defined by the list $(x_1, \ldots, x_m)$ and let $\tau = C(F) \cap -C(G)$ for faces F and G of P.

First, we consider $C(v)$ for a vertex v of P. Suppose that $T_{i_1}, \ldots, T_{i_\ell}$ is the list of hyperplanes such that $T_{i_j} = e_j^\perp$ for an edge e_j containing v.[6] A linear functional f on P is maximized at v if and only if $f(v) \geq f(w)$ for all vertices w adjacent to v.[7] Let edge e_j containing v have endpoints v and v_j. Then $C(v)$ is the set of all $a \in V$ such that $a \cdot v \geq a \cdot v_j$ for all v_j. For any single j, the set of these a is exactly $T_{i_j}^{x_{i_j}}$ where $x_{i_j} = \pm 1$. Hence, we can write

$$C(v) = T_{i_1}^{x_{i_1}} \cap \cdots \cap T_{i_\ell}^{x_{i_\ell}}$$

where $(x_{i_1}, \ldots, x_{i_\ell})$ is a list of entries from $\{+1, -1\}$. We now turn our attention back to a general face F of P. Suppose that F has vertices $v_1, \ldots, v_k$. A linear functional is maximized on F if and only if it is maximized on all vertices of F. Hence, we have

$$C(F) = C(v_1) \cap \cdots \cap C(v_k).$$

Each $C(v_i)$ can be written as the intersection of spaces $T_j^{x_j}$ using the process above. Suppose for vertices v and w of F, there is some j $(1 \leq j \leq m)$ such that $T_j^{x_j}$ appears in the intersection for $C(v)$ and $T_j^{y_j}$ appears in the intersection for $C(w)$. Then $T_j^{x_j} \cap T_j^{y_j} = T_j^{z_j}$ where $z_j = x_j$ if $x_j = y_j$ and $z_j = 0$ if $x_j \neq y_j$. Using this reasoning, we conclude that $C(F)$ can be written as the intersection of spaces $T_{i_j}^{z_{i_j}}$ for $z_{i_j} \in \{-1, 0, +1\}$.

$$C(F) = T_{i_1}^{z_{i_1}} \cap \cdots \cap T_{i_p}^{z_{i_p}}$$

This same process can be used to rewrite $-C(G)$ as the intersection of such spaces; suppose that G has vertices $u_1, \ldots, u_{k^*}$. We use two facts analogous to those above but applied to linear functionals minimizing faces on P. A linear functional f is minimized at a vertex v if and only if $f(v) \leq f(w)$ for all vertices w adjacent to v. Also, a linear functional is minimized on G if and only if it is minimized on all vertices of G. Hence, we can write $-C(G)$ as follows.

$$\begin{aligned} -C(G) &= -C(u_1) \cap \cdots \cap -C(u_{k^*}) \\ &= T_{j_1}^{w_{j_1}} \cap \cdots \cap T_{j_q}^{w_{j_q}} \end{aligned}$$

We would like to show that $\sigma \cap \big(C(F) \cap -C(G)\big)$ is a face of $\Sigma(P)$. Consider only $\sigma \cap C(F)$.

$$\begin{aligned} \sigma \cap C(F) &= \sigma \cap \big(T_{i_1}^{z_{i_1}} \cap \cdots \cap T_{i_p}^{z_{i_p}}\big) \\ &= \bigcap_{i=1}^{m} T_i^{y_i} \quad \text{where } y_i = \begin{cases} 0 & \text{if } i = i_j \text{ for some } 1 \leq j \leq p \text{ and } x_i \neq z_{i_j} \\ x_i & \text{otherwise} \end{cases} \end{aligned}$$

Hence, we see that $\sigma \cap C(F)$ is a face of $\Sigma(P)$. Then we can rewrite

$$\sigma \cap \big(C(F) \cap -C(G)\big) = \big(\sigma \cap C(F)\big) \cap -C(G)$$

and, since $-C(G)$ can be written as the intersection of spaces $T_j^{w_j}$, we see that this intersection is a face of $\Sigma(P)$ by applying the same reasoning. Therefore, $\Sigma(P)$ is a refinement of $\Gamma(P)$. □

[6] If two edges share a vertex, they cannot be parallel. Hence, there will be one T_{i_j} for each edge e_j.

[7] This follows from the convexity of P.

Earlier we discussed *generic* linear functionals: those which are non-constant on all k-faces of P for $k \geq 1$. Notice that if $a \in V$ defines a generic linear functional, then a cannot lie in $e^\perp$ for any edge e of P. Hence the generic linear functionals are exactly those in the n-dimensional faces of $\Sigma(P)$.

Lemma 3.10 in the next section proves that the faces of $\Sigma(P)$ completely determine the distinct linear functionals. Suppose f_1 and f_2 are linear functionals with associated vectors a_1 and a_2. If a_1 and a_2 lie in the same face of $\Sigma(P)$, then the Baues posets $\omega(P, f_1)$ and $\omega(P, f_2)$ are identical.

Consider instead if f_1 and f_2 are generic and a_1 and a_2 lie in different n-faces of $\Sigma(P)$ (different connected components of $V \setminus \bigcup e^\perp$). As we rotate a_1 continuously to a_2, there will be a moment where we pass through some hyperplane $e^\perp$, corresponding to a non-generic linear functional. In order to connect the Baues posets from the starting and ending positions, we must define the Baues poset for this non-generic linear functional. In the next section, we will expand all the definitions from Section 2 to include non-generic linear functionals.

3.2 Including non-generic linear functionals

We start by defining cellular strings in generality.

Definition 3.5. Let P be a polytope in $\mathbb{R}^n$. A list of faces of P given by $\mathcal{F} = (F_0, F_1, \ldots, F_{k-1}, F_k)$ is a *cellular string* if there is a linear functional $f : \mathbb{R}^n \to \mathbb{R}$ ($f \not\equiv 0$) such that

(i) F_0 is the face minimizing f on P;

(ii) F_k is the face maximizing f on P;

(iii) $F_0 \cap F_1 \neq \emptyset$ and $F_{k-1} \cap F_k \neq \emptyset$;

(iv) f is non-constant on all faces F_i for $1 \leq i \leq k-1$;

(v) the f-maximum face of F_i is the f-minimum face of F_{i+1} for $1 \leq i \leq k-2$.

Conditions (i) and (ii) in Definition 3.5 are analogous to condition (i) in Definition 1.4. Conditions (iv) and (v) in Definition 3.5 are exactly conditions (ii) and (iii) in Definition 1.4. Lastly, condition (iii) of Definition 3.5 ensures that the cellular string will be connected. Sometimes we refer to a cellular string as only a list of faces apart from any specific linear functional. This is helpful when comparing cellular strings. Other times, when the linear functional is relevant, we will call $\mathcal{F}$ an f-cellular string.

If we fix a generic linear functional f, then this definition agrees with Definition 1.4 except that the cellular strings will have extra "tags": $v_{\min}$ appended at the beginning and $v_{\max}$ appended at the end. As we begin to look at cellular strings from many different linear functionals at once, these tags indicate the direction of the cellular string.

Consider the example in Figure 10. See that $\mathcal{F}_1 = (v_0, F, v_1)$ is an f_1-cellular string and $\mathcal{F}_2 = (v_1, F, v_0)$ is an f_2-cellular string. We notice that $\mathcal{F}_1$ and $\mathcal{F}_2$ consist of the same set of faces (namely $\{F, v_0, v_1\}$), but the *direction* from minimum to maximum according to each linear functional is different. Hence, we would like to consider $\mathcal{F}_1$ and $\mathcal{F}_2$ as different cellular strings. Thus, the "tags" of the minimum and maximum faces indicate the direction of a cellular string, even when it consists of only one face. On the other hand, $\mathcal{F}_1$ is also an f_3-cellular string. Notice that f_1 and f_3 are in the same face of $\Sigma(P)$. We will see in Lemma 3.10 that f_1 and f_3 have identical sets of cellular strings.

Now that we have defined cellular strings for any linear functional, we would like to be able to compare cellular strings derived from different linear functionals.

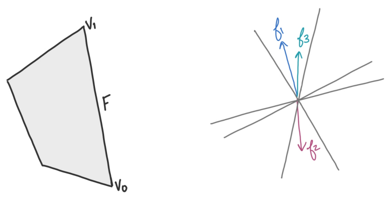


Figure 10: On the left is the polytope P with faces labeled v_0, F, and v_1. On the right is $\Sigma(P)$ with three linear functionals. This shows a single collection of faces forms different cellular strings depending on the linear functional.

Definition 3.6. For two cellular strings $\mathcal{F} = (F_0, F_1, \ldots, F_{k-1}, F_k)$ and $\mathcal{E} = (E_0, E_1, \ldots, E_{m-1}, E_m)$, we say that $\mathcal{F} \leq \mathcal{E}$ if and only if

(i) $\bigcup_{i=0}^{k} F_i \subseteq \bigcup_{i=0}^{m} E_i$,

(ii) $F_0 \subseteq E_0$ and $F_k \subseteq E_m$.

Compare Definition 3.6 with the partial order we assigned to cellular strings from the same generic linear functional in Section 2. The extra condition (ii) of Definition 3.6 is exactly because of the issue with direction which we were exploring above. We only want to compare two cellular strings if they are in the same direction from minimum to maximum.

With the definition for a general cellular string and a way to compare cellular strings across linear functionals, we can now define a mega Baues poset consisting of all cellular strings on a polytope.

Definition 3.7. Let P be an n-dimensional polytope in $\mathbb{R}^n$. The *mega Baues poset* of P, denoted $\Omega(P)$ is the set of all cellular strings on P with the partial order given in Definition 3.6.

Remark. *Note that we have defined $\Omega(P)$ such that each cellular string only appears once, even though it can be an f-cellular string for many different linear functionals f. Hence, since P is finite, we can be sure that $\Omega(P)$ is as well.*

3.3 Creating a cell complex of linear functionals

We now return to our organization of all linear functionals on a polytope from Section 3.1. We know that some linear functionals generate identical sets of cellular strings, and hence identical Baues posets. For example, $f(x)$ and $c \cdot f(x)$ for a positive constant c as we discussed at the end of Section 1. Additionally, if we take a generic linear functional f and wiggle it a little to get f^*, intuition tells us that all f-cellular strings will also be f^*-cellular strings. To completely categorize all linear functionals on P, we will take the polyhedral fan $\Sigma(P)$ defined in Section 3.1 and use it to construct a *regular cell complex* of linear functionals.

An *open cell* is a topological space homeomorphic to the open unit ball $\mathbb{B}^n$. A *closed cell* is a topological space homeomorphic to the closed unit ball $\overline{\mathbb{B}^n}$. We call these n-dimensional cells or just an n-cells.

Definition 3.8. Let X be a nonempty topological space. A *cell decomposition* of X is a partition $\mathscr{E}$ of X into open cells. For each $e \in \mathscr{E}$ of dimension $n \geq 1$, there is a continuous map Φ from some closed n-cell D into X such that Int D is homeomorphic to e under Φ and $\Phi(\partial D)$ is contained in the union of cells of $\mathscr{E}$ with dimension strictly less than n.

The map Φ for cell e is called its *characteristic map.* If X is Hausdorff, the space X with a cell decomposition is a *cell complex.* A CW complex is a cell complex $(X, \mathscr{E})$ satisfying the following conditions:

1. The closure of each cell is contained in a union of finitely many cells.

2. $U \subseteq X$ is open if and only if $U \cap \overline{e}$ is open in $\overline{e}$ for each $e \in \mathscr{E}$.

The dimension of X is the largest n such that X contains at least one n-cell. An open cell e is called a *regular cell* if its characteristic map is a homeomorphism onto $\overline{e}$. That is, if $e \cong \mathbb{B}^n$ then $\overline{e} \cong \overline{\mathbb{B}^n}$ with the same homeomorphism. A CW complex is a *regular CW complex* (or just a *regular cell complex*) if each of its cells is regular and the closure of each cell is a finite subcomplex.

Regular cell complexes will be useful because they have properties which we would like to expect of our topological spaces. For example, they have the following nice property:

Let X be a regular cell complex. Then if Γ is the face poset of X, the order complex on Γ is homeomorphic to X.

We can now turn back to $\Sigma(P)$ (for a n-dimensional polytope P) which we were studying in Section 3.1. At the end of Section 1, we saw that the magnitude of a linear functional has no effect on the creation of cellular strings, and thereby the creation of the Baues poset. We will then only consider linear functionals with associated vector a of unit length ($a \in \mathbb{S}^{n-1} \subseteq V$). Then we can use $\Sigma(P)$ to give $\mathbb{S}^{n-1}$ a regular cell decomposition by intersecting each open face of $\Sigma(P)$ with $\mathbb{S}^{n-1}$. We will let $M(P)$ refer to $\mathbb{S}^{n-1}$ along with the cell decomposition inherited from $\Sigma(P)$.[8] In fact, this process works for any polyhedral fan as stated in Lemma 3.9. It follows from the fact that the faces of a polyhedral fan are convex polyhedral cones and they have the same intersection properties required of the closed cells of a regular cell decomposition.

Lemma 3.9. *If $\mathcal{K}$ is a polyhedral fan covering $\mathbb{R}^n$, then $\mathscr{E} := \{\sigma \cap \mathbb{S}^{n-1} \mid \sigma$ is an open face of $\mathcal{K}\}$ is a regular cell decomposition of $\mathbb{S}^{n-1}$.*

The faces of a convex polytope are also convex polytopes of smaller dimension. We can then consider a linear functional on P as a linear functional on a face F of P. For any face $F \subseteq P$ of dimension at least 1, let S be the smallest affine subspace of $\mathbb{R}^n$ that contains F. Then

$$\dim(S) = \dim(F) =: m.$$

Let $T \subseteq V$ be a translation of S that contains the origin. See that T is a linear subspace of V so it is a copy of $\mathbb{R}^m$. We can then view it as the dual space of F consisting of all linear functionals on this face. For any vector $a \in V$, define $a_{\|}$ to be the projection of a onto T.[9] Then $a = a_{\|} + a_{\perp}$ where $a_{\perp}$ is a vector perpendicular to T. Since T is parallel to S and $F \subseteq S$, we know that $a_{\perp}$ is also perpendicular to F. Thus $a_{\perp} \cdot x$ is constant for all $x \in F$: the linear functionals constant on a face are precisely those with associated vectors orthogonal to that face (recall Section 3.1). We can write

$$a \cdot x = a_{\|} \cdot x + a_{\perp} \cdot x$$

using distributive properties of the dot product. If a is non-constant on F, then a is not perpendicular to F so $a_{\|}$ is not the zero vector. Hence, we conclude that $a \cdot x$ is non-constant on a face of F if and only if $a_{\|}$ is. We will use this discussion in the proof of the following lemma.

[8]'M' stands for map!

[9]If $v_1, \ldots, v_m$ are a basis for T, then $a_{\|} = \sum_{i=1}^{m} \operatorname{proj}_{v_i} a$.

Lemma 3.10. *Let f_1 and f_2 be linear functionals on n-dimensional polytope $P \subset \mathbb{R}^n$ with associated vectors a_1 and a_2 in $M(P)$. If a_1 and a_2 lie in the same open cell of $M(P)$, then $\omega(P, f_1) = \omega(P, f_2)$.*

Proof. To prove this, we will show that every f_1-cellular string is also an f_2-cellular string. Then we can apply the same logic to f_2 to show that these linear functionals generate the same set of cellular strings and thus have identical Baues posets.

Suppose that $\mathcal{F} = (F_0, F_1, \ldots, F_k)$ is an f_1-cellular string and we will show $\mathcal{F}$ is also an f_2-cellular string using the conditions in Definition 3.5 on page 11. First, we recall that $\Sigma(P)$ is a refinement of $\Gamma(P)$ by Lemma 3.4. If a_1 and a_2 lie in the same cell of $M(P)$, then they also lie in the same face of $\Sigma(P)$. Hence, they are in the same face of $\Gamma(P)$ and must have the same maximum and minimum faces over P. Thus, conditions (i) and (ii) are satisfied.

Condition (iii) ($F_0 \cup F_1 \neq \emptyset$ and $F_{k-1} \cap F_k \neq \emptyset$) is independent of the linear functional, so this will be satisfied. Since f_1 and f_2 lie in the same open cell of $M(P)$, by construction they are constant on the same faces of P. Hence, f_2 must be non-constant on all faces F_i of $\mathcal{F}$ (for $1 \leq i \leq k-1$) so condition (iv) is satisfied.

Lastly, we show that $\mathcal{F}$ satisfies condition (v) of Definition 3.5 with respect to f_2. As already discussed, a_1 and a_2 are non-constant on all faces $F_i \in \mathcal{F}$ for $1 \leq i \leq k-1$. For any face F_i, let $a_{1\parallel}$ and $a_{2\parallel}$ be vectors in $M(F_i)$ as constructed in the discussion before this lemma. We know a_1 and a_2 are constant on the same set of faces of F_i since they are in the same cell of $M(P)$. This implies that $a_{1\parallel}$ and $a_{2\parallel}$ will also be constant on the same set faces of F_i and so $a_{1\parallel}$ and $a_{2\parallel}$ are in the same cell of $M(F_i)$. By applying the argument above for conditions (i) and (ii), we conclude that $a_{1\parallel}$ and $a_{2\parallel}$ have the same maximum and minimum faces on F_i. We write (for $j = 1, 2$)

$$a_j \cdot x = a_{j\parallel} \cdot x + a_{j\perp} \cdot x.$$

Since $a_{j\perp} \cdot x$ is constant on F_i, we see that a_j will be maximized (or minimized) over the same face of F_i as $a_{j\parallel}$. Therefore, a_1 and a_2 also have the same maximum and minimum faces over F_i. Hence, $\mathcal{F}$ satisfies condition (v) with respect to f_2.

By Definition 3.5, this shows that $\mathcal{F}$ is an f_2-cellular string as well. This argument holds for all f_1-cellular strings so $\omega(P, f_1) \subseteq \omega(P, f_2)$. We can apply the same argument to all f_2-cellular strings to get the other inclusion and so $\omega(P, f_1) = \omega(P, f_2)$. □

With this Lemma, we see that $M(P)$ categorizes all distinct linear functionals on P. Additionally, $M(P)$ is a *finite* cell complex because there are only finitely many edges of P so only finitely many hyperplanes in $\Sigma(P)$. Hence, we have reduced the infinitely many linear functionals on P down to only a finite list. This combinatorial structure of linear functionals will play a large part in the process of proving the homotopy type of $\Omega(P)$.

3.4 Understanding our goals and the remaining tools to get there

Consider the mega Baues poset from a high-level perspective. Choosing a linear functional on P is equivalent to choosing a point in $\mathbb{S}^{n-1}$. Then look at P from the perspective of its minimum face under this linear functional. A cellular string will start by taking one step away from the minimum faces along any of its adjacent faces. This creates a "circle of possibilities" around the minimum face. In effect, we are choosing a point on $\mathbb{S}^{n-1}$ and then picking a direction along the surface of $\mathbb{S}^{n-1}$ to travel away from that point. We will see that $\Omega(P)$ *is* this space, up to homotopy. This intuitive idea is made rigorous with the notion of the *unit tangent bundle* of a manifold.

Let M be a Riemannian manifold. For each point $x \in M$, the tangent space at x to M is denoted $T_x(M)$. A vector $v \in T_x(M)$ has unit length if it has unit length with respect to the metric inherited by $T_x(M)$.

(a) The polytope P. (b) The projection $\pi(P)$.

Figure 11: A polytope P with linear functional f given by vector a. We choose vector b orthogonal to a and project P onto the plane spanned by these vectors using the projection π. Cellular strings on the polygon $\pi(P)$ correspond to coherent cellular strings on P.

Definition 3.11. The *unit tangent bundle* on M (denoted $UT(M)$) is the set of pairs (x, v) where $x \in M$ and $v \in T_x(M)$ has unit length.

There is a natural projection $\pi : UT(M) \to M$ where $\pi(x, v) = x$. The fibers $\pi^{-1}(x)$ of this map are homotopic to $\mathbb{S}^{k-1}$ where k is the dimension of M. In particular, we will focus on the unit tangent bundle on the sphere. For a sphere $\mathbb{S}^k$, the unit tangent bundle $UT(\mathbb{S}^k)$ is bijective to $\mathbb{S}^k \times \mathbb{S}^{k-1}$ by what we have said about the fibers of π above. However, it has a more interesting topology than that which this space inherits from the ambient Euclidean space.

Conjecture 3.12. *For an n-dimensional polytope P in $\mathbb{R}^n$, the mega Baues poset $\Omega(P)$ is homotopy equivalent to the unit tangent bundle on $\mathbb{S}^{n-1}$.*

For an intuitive picture of this homotopy equivalence, notice that each Baues poset $\omega(P, f)$ is homotopy equivalent to $\mathbb{S}^{n-2}$ by Theorem 2.2. Then we place these around $M(P)$ (which is a cell decomposition of $\mathbb{S}^{n-1}$), one Baues poset for each cell of the complex. The connections in $\Omega(P)$ across the Baues posets will be "nice enough" to make this space homotopy equivalent to $UT(\mathbb{S}^{n-1})$.

Let us go back to the picture we were discussing at the beginning of this section. For each linear functional f given by $a \in \mathbb{S}^{n-1} \subseteq V$, we choose an orthogonal vector to a which determines the direction along the surface of $\mathbb{S}^{n-1}$. This orthogonal vector b also lives in V and so it defines a linear functional. We claim that b determines a unique f-cellular string $\mathcal{F}_b$.[10]

First we project P onto the plane spanned by vectors a and b, letting π denote this projection. Figure 11 shows this process for a three-dimensional polytope. If $F_{\max}$ is the maximum face of $\pi(P)$ under a, then $\pi^{-1}(F_{\max})$ is the maximum face of P under a, and similarly for the minimum face $F_{\min}$. The polygon $\pi(P)$ has two f-cellular strings, one on the 'b side' and one on the '$-b$ side' as depicted in Figure 11b. Let $\mathcal{E}_b = (E_0, E_1, \ldots, E_{k-1}, E_k)$ be the cellular string in the b direction. Then $\mathcal{F}_b = (F_0, \ldots, F_k)$ where $F_i = \pi^{-1}(E_i)$ is a cellular string on P.[11] This is what we call a coherent cellular string.

Definition 3.13. An f-cellular string $\mathcal{F} = (F_0, F_1, \ldots, F_{k-1}, F_k)$ is *coherent* if there is another

[10] That is, there is only one $\mathcal{F}_b$ for each b but multiple choices for b can give the same cellular string.
[11] There a few missing details here, but we leave it to the reader to check.

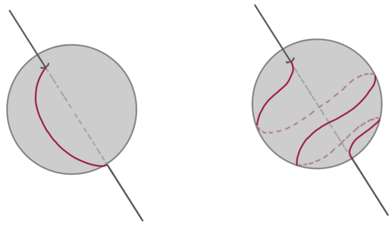


Figure 12: A path representing a coherent cellular string (left) and a path representing a non-coherent cellular string (right).

linear functional $g : \mathbb{R}^n \to \mathbb{R}$ such that

$$\bigcup_{i=1}^{k-1} F_i = \{p \in P \mid g(p) \geq g(x) \text{ for all } x \in f^{-1}(f(p))\}.$$

We can think of coherent cellular strings as being those which stay on one "side of the polytope" as it moves from minimum to maximum. Imagine taking a skewer and sticking the polytope from minimum to maximum like a kabob. A coherent cellular string would stay somewhat parallel to the skewer whereas a non-coherent cellular string would wrap around the kabob. See Figure 12. This can only be visualized in three-dimensions, but it gives us a geometric sense for the formal definition.

The subset of the mega Baues poset consisting of only coherent cellular strings is in direct correspondence with the unit tangent bundle. For each linear functional f, we choose some other linear functional g and this defines a coherent cellular string. Without loss of generality, we can choose g to be orthogonal to f with unit length.[12] The set of all choices for f and g is exactly the unit tangent bundle. If we partition pairs (f, g) based on the resulting cellular string, we have a regular cell decomposition of $UT(\mathbb{S}^{n-1})$. In this way, the set of coherent cellular strings is a combinatorial version of the unit tangent bundle.

The particular use of coherent cellular strings in our investigation is implicit rather than explicit. In the following section, we will ultimately prove our claim by deforming each cellular string into a coherent cellular string. Imagine a sphere punctured by a skewer through its center. For any simple path from one puncture point to the other, we can continuously slide the path along the surface of the sphere, deforming it so it becomes a great circle. This great circle on the sphere is analogous to a coherent cellular string. Our approach to proving the claim will fill in the details of this high-level approach.

4 $\Omega(P)$ is homotopy equivalent to $UT(\mathbb{S}^{n-1})$

The mega Baues poset is a *combinatorial* object which tends to have lots of details and be challenging to describe. In order to continue working with it, we will translate the mega Baues poset into an indiscrete object which does not necessarily depend on the geometry of the polytope. The mega Baues poset is essentially the set of all discrete paths (cellular strings) on the boundary of a polytope.

[12] Because for any choice of g there is some g' orthogonal to f with unit length such that g and g' give the same cellular string.

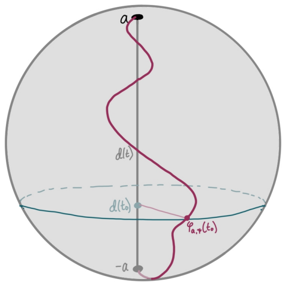


Figure 13: A pair (a, ψ) in Λ^n defines a path $\varphi_{a,\psi}$ on the sphere $\mathbb{S}^{n-1}$. The point $\varphi_{a,\psi}(t_0)$ for some $t_0 \in [0, 1]$ in constructed here.

The boundary of an n-dimensional polytope is homotopy equivalent to the $n-1$-dimensional sphere; this leads us to reinterpret the mega Baues poset as continuous paths on the sphere.

4.1 Translating the mega Baues poset

Let Λ^n be the set of all pairs (a, ψ) such that $a \in \mathbb{S}^{n-1} \subseteq \mathbb{R}^n$ and $\psi : [0, 1] \to \mathbb{S}^{n-2}$ is a continuous map. We can view these pairs as paths on the sphere $\mathbb{S}^{n-1}$ between antipodal points. For a pair (a, ψ), we define a path $\varphi_{a,\psi} : [0, 1] \to \mathbb{S}^{n-1}$ in the following way.

Let $d : [0, 1] \to \mathbb{R}^n$ be the unit speed parameterization of the diameter of $\mathbb{S}^{n-1}$ from $-a$ to a such that $d(0) = -a$ and $d(1) = a$. Define $\varphi_{a,\psi}(0) = -a$ and $\varphi_{a,\psi}(1) = a$. Then for $t \in (0, 1)$, the intersection of $\mathbb{S}^{n-1}$ with the hyperplane orthogonal to the diameter at $d(t)$ forms a copy of $\mathbb{S}^{n-2}$. We let this $\mathbb{S}^{n-2}$ inherit its coordinate directions from the ambient space $\mathbb{R}^n$. Then $\psi(t)$ determines a point on $\mathbb{S}^{n-2}$ which we designate $\varphi_{a,\psi}(t)$. Since d and ψ are continuous, $\varphi_{a,\psi}$ is as well. Figure 13 displays an example of constructing the path $\varphi_{a,\psi}$ in the three-dimensional case.

Recall the normal fan $N(P)$ for an n-dimensional polytope P in $\mathbb{R}^n$. By Lemma 3.9, $N(P)$ gives $\mathbb{S}^{n-1}$ a regular cell decomposition. Each open cell in the cell decomposition is associated to a unique face on P that is maximized by all linear functionals in that cell.

Now we define $\Phi : \mathbb{S}^{n-1} \to \{\text{faces of } P\}$ where we view v as a vector in $\mathbb{R}^n$ and $\Phi(v)$ is the face of P maximized by the linear functional v. Then, for a pair (a, ψ), we define its *face bundle* by

$$\Delta(a, \psi) := \Phi\big(\varphi_{a,\psi}([0, 1])\big).$$

The face bundle is simply a set of faces of the polytope P. Ultimately, we would like to determine when this set of faces forms a cellular string, but a general $\Delta(a, \psi)$ already has some of the necessary structure. Since $\varphi_{a,\psi}$ is continuous and increasing along the diameter from $-a$ to a, we know $\Delta(a, \psi)$ forms a connected path starting at the minimum face according to a and ending at the maximum face according to a.

We can give the set of all face bundles a partial order. For pairs (a, ψ_1) and (b, ψ_2), we say that $\Delta(a, \psi_1) \leq \Delta(b, \psi_2)$ if

$$\bigcup_{F \in \Delta(a,\psi_1)} F \subseteq \bigcup_{E \in \Delta(b,\psi_2)} E, \quad \Phi(\varphi_{a,\psi_1}(0)) \subseteq \Phi(\varphi_{b,\psi_2}(0)) \quad \text{and} \quad \Phi(\varphi_{a,\psi_1}(1)) \subseteq \Phi(\varphi_{b,\psi_2}(1)).$$

There are conditions beyond simply connectedness and starting/ending positions that we require for cellular strings. In order to determine when $\Delta(a, \psi)$ has these properties, we can place extra requirements on (a, ψ). Understanding exactly which (a, ψ) give cellular strings will be a key step in translating the mega Baues poset into an indiscrete object.

First, we have a few definitions to make things easier. The span of a cell e of $\mathbb{S}^{n-1}$ is the span in $\mathbb{R}^n$ of all vectors in that cell. We write this subspace as $\operatorname{span}(e)$. We claim that if $a \in \operatorname{span}(e)$, then a is constant on the face maximized by linear functionals in e. Let F be the face maximized by linear functionals in e. Then $e = C(F) \cap \mathbb{S}^{n-1}$ and also $\operatorname{span}(C(F)) = F^{\perp}$ because $C(F)$ is a full-dimensional subset of $F^{\perp}$. Hence, if $a \in \operatorname{span}(e)$, then $a \in \operatorname{span}(C(F)) = F^{\perp}$ which means that a is constant on F.

Next, suppose that $\varphi_{a,\psi}(t_0) \in e$ for $t_0 \in (0, 1)$. We assume that $a, -a \notin e$ (so e is not the "first" or "last" cell crossed by $\varphi_{a,\psi}(t)$). We let $\varepsilon > 0$ be small. The *cell before* e is the open cell e' containing $\varphi_{a,\psi}(t_0 - \varepsilon)$ such that $e' \neq e$ and ε is minimized. The *cell after* e is the open cell e'' containing $\varphi_{a,\psi}(t_0 + \varepsilon)$ such that $e'' \neq e$ and ε is minimized.

For a fixed t_0 (with the same conditions as above) we define the *straight path at* $\varphi_{a,\psi}(t_0)$ to be the path given by $(a, c) \in \Lambda^n$ where c is the constant map $c(t) = \psi(t_0)$ for all t. Then notice that $\varphi_{a,c}(t)$ is a great circle on $\mathbb{S}^{n-1}$ passing through the points a, $\varphi_{a,\psi}(t_0)$, and $-a$. If $\varphi_{a,\psi}(t_0) \in e$ and $a \notin \operatorname{span}(e)$, then we define the *cell below* e to be the cell before e according to $\varphi_{a,c}(t)$. Similarly, the *cell above* e is the cell after e according to $\varphi_{a,c}(t)$. If e^- is the cell below e and e^+ is the cell above e, then $\Phi(e^-)$ is the minimum face of $\Phi(e)$ according to a and $\Phi(e^+)$ is its maximum face.[13]

Definition 4.1. Given an n-dimensional polytope P, let $\mathbb{S}^{n-1}$ have the regular cell decomposition given by $N(P)$. The *cellular string complex* of P, denoted by $\mathcal{C}(P)$, is a subset of Λ^n consisting of all pairs (a, ψ) satisfying the following conditions:

(i) For each open cell e of $\mathbb{S}^{n-1}$, $\varphi_{a,\psi}^{-1}(e) \subseteq [0, 1]$ is connected.

(ii) If there is some t such that $\varphi_{a,\psi}(t) \in e$ and $a \notin \operatorname{span}(e)$, then the cell before e is the cell below e and the cell after e is the cell above e.

(iii) If there is some t such that $\varphi_{a,\psi}(t) \in e$ and $a \in \operatorname{span}(e)$, then a is not in the span of the cell before e or the cell after e (where these are defined).

We will show that the face bundle of any pair in $\mathcal{C}(P)$ is essentially a cellular string. To do this, we need a canonical way to generate a list of faces from such a face bundle. Suppose that $(a, \psi) \in \mathcal{C}(P)$. From this point forward will write φ in place of $\varphi_{a,\psi}$ except when the pair (a, ψ) is unclear.

We know that $\varphi(t)$ passes through only finitely many cells because there are finitely many faces on the polytope. Let $\sigma_0, \ldots, \sigma_k$ be an arbitrary ordering of the faces in $\Delta(a, \psi)$. Then $\Phi^{-1}(\sigma_i)$ is an open cell in $\mathbb{S}^{n-1}$. Define

$$I_i := \varphi^{-1}\big(\Phi^{-1}(\sigma_i)\big).$$

Note that I_i is a connected subset of $[0, 1]$ because of condition (i) in Definition 4.1 so it must be a (possibly degenerate) interval. If $i \neq j$, then I_i and I_j must be disjoint by definition. Also, every $t \in [0, 1]$ must be in some I_i since $\Phi\big(\varphi(t)\big) = \sigma_i$ for some i. Therefore, $\{I_i\}_{i=0}^{k}$ forms a disjoint covering of $[0, 1]$ and we can reorder this set of intervals to be ascending. Relabel these $I_0, \ldots, I_k$ so that $I_i \leq I_{i+1}$ for all i.[14]

[13]This requires some proof, but it amounts to looking at the geometry of the vectors contained in e, e^-, and e^+.

[14]That is, $t_i \leq t_{i+1}$ for all $t_i \in I_i$ and $t_{i+1} \in I_{i+1}$.

We start by defining a list $\mathcal{F}^*$. This will order the faces of $\Delta(a, \psi)$ in the desired way, but it is not quite a cellular string because it will contain all of the "in between" faces as well.

$$\mathcal{F}^* := (F_0^*, \dots, F_k^*) \quad \text{where } F_i^* := \Phi\big(\varphi(I_i)\big) \text{ for } 0 \le i \le k$$

Then we create a list $\mathcal{F}$ from $\mathcal{F}^*$ by deleting all the unnecessary faces but keeping the order designated by $\mathcal{F}^*$. If $F_i^* \subseteq F_{i+1}^*$, then delete F_i^* from the list. If $F_{i+1}^* \subseteq F_i^*$, then delete F_{i+1}^* from the list. Repeat this process iteratively until we are left with a list of faces where no two adjcacent faces are comparable. We relabel these $F_0, \dots, F_m$, keeping the order and we define $\mathcal{F} = (F_0, \dots, F_m)$.

Theorem 4.2. *If $(a, \psi) \in \mathcal{C}(P)$, then the list $\mathcal{F}$ obtained by the process described above is a cellular string.*

Lemma 4.3. *In the list $\mathcal{F}^*$, two adjacent faces F_i^* and F_{i+1}^* are comparable.*

Proof of Lemma 4.3. By construction of the intervals, we know that $J := I_i \cup I_{i+1}$ is an interval. Also, $\varphi(J)$ is contained in exactly two cells of $\mathbb{S}^{n-1}$ (with cell decomposition from $N(P)$). Since φ is a continuous map and $\varphi(J)$ must be a connected set, these cells must be comparable, implying that F_i^* and F_{i+1}^* are comparable. $\square$

Proof of Theorem 4.2. We will show that $\mathcal{F} = (F_0, \dots, F_m)$ is a cellular string with respect to the linear functional associated to vector a. To do this, we check all conditions in Definition 3.5.

First, we know that $F_0 = \Phi(\varphi(0)) = \Phi(-a)$ so F_0 is, by definition of Φ, the face maximizing $-a$ on P. Hence, it is also the face minimizing a on P. In the same way, $F_m = \Phi(\varphi(1)) = \Phi(a)$ is the face maximizing a on P. Thus, conditions (i) and (ii) are satisfied.

Next, to show that $\mathcal{F}$ satisfies condition (iii), we prove that for any F_i and F_{i+1} in $\mathcal{F}$, we have $F_i \cap F_{i+1} \neq \emptyset$. Note that F_i and F_{i+1} are not comparable by construction of $\mathcal{F}$ and also $F_i = F_{j_1}^*$ and $F_{i+1} = F_{j_2}^*$ for faces $F_{j_1}^*$ and $F_{j_2}^*$ ($j_1 < j_2$) of the list $\mathcal{F}^*$. By assumption, all faces F_i^* for $j_1 < i < j_2$ are deleted by the process which creates $\mathcal{F}$. Hence, each F_i^* must be contained in at least one of F_{i-1}^* or F_{i+1}^*. This implies that there is some $F_{i_0}^*$ contained in both $F_{j_1}^*$ and $F_{j_2}^*$. Thus, $F_i \cap F_{i+1} \neq \emptyset$.

Condition (iv) of Definition 3.5 requires that a is nonconstant on F_i for all $1 \le i \le m-1$. Suppose that there is some F_i^* in the list $\mathcal{F}^*$ on which a is constant. Then we claim that F_i^* will be deleted in the creation of the list $\mathcal{F}$. Suppose for contradiction that F_i^* is not deleted so

$$F_i^* \not\subseteq F_{i+1}^* \quad \text{and} \quad F_i^* \not\subseteq F_{i-1}^*.$$

Since adjacent faces in $\mathcal{F}^*$ must be comparable, we then have $F_{i+1}^* \subseteq F_i^*$ and $F_{i-1}^* \subseteq F_i^*$. However, this implies that

$$\Phi^{-1}(F_i^*) \subseteq \Phi^{-1}(F_{i+1}^*) \quad \text{and} \quad \Phi^{-1}(F_i^*) \subseteq \Phi^{-1}(F_{i-1}^*).$$

Because a is constant on F_i^*, we know that $a \in \operatorname{span}(\Phi^{-1}(F_i^*))$. However, the inclusion above implies that $a \in \operatorname{span}(\Phi^{-1}(F_{i+1}^*))$ and $a \in \operatorname{span}(\Phi^{-1}(F_{i-1}^*))$ as well. This contradicts condition (iii) of Definition 4.1 because $\Phi^{-1}(F_{i-1}^*)$ is the cell before $\Phi^{-1}(F_i^*)$ and $\Phi^{-1}(F_{i+1}^*)$ is the cell after $\Phi^{-1}(F_i^*)$. Hence, we conclude that the assumption was wrong and F_i^* must be deleted in the construction of $\mathcal{F}$.

Lastly, we show that $\mathcal{F}$ satisfies condition (v): the face maximizing a on F_i is the face minimizing a on F_{i+1} for $1 \le i \le m-2$. Since F_i and F_{i+1} are in the list $\mathcal{F}$, we know that a is nonconstant on these faces as we argued above. This implies that a is not contained in $\operatorname{span}(\Phi^{-1}(F_i))$ or $\operatorname{span}(\Phi^{-1}(F_{i+1}))$. Let $F_i = F_{j_1}^*$ and $F_{i+1} = F_{j_2}^*$ for faces $F_{j_1}^*$ and $F_{j_2}^*$ of $\mathcal{F}^*$. By condition (ii) of Definition 4.1, we know that $E^+ := F^*_{j_1+1}$ is the maximum face of F_i and $E^- := F^*_{j_2-1}$ is the minimum face of F_{i+1}. Suppose for contradiction that $E^+ \neq E^-$.

First, we consider any cell e such that $\varphi_{a,\psi}(t_0) \in e$ but $a \notin \text{span}(e)$. Let $\varphi_{a,c}(t)$ be the straight path at $\varphi_{a,\psi}(t_0)$. Since $\varphi_{a,c}(t)$ is a great circle passing through a, any open subset of this path will have a span containing a. Thus, e must not contain any open subset of $\varphi_{a,c}(t)$ so $\varphi_{a,c}(t) \cap e$ is a single point: $\varphi_{a,c}(t_0) = \varphi_{a,\psi}(t_0)$. Since $\varphi_{a,c}(t)$ is a connected continuous path, we see that the cell below e and the cell above e must contain an open subset of $\varphi_{a,c}(t)$. Hence a is contained in the span of the cell below/above e.

By this argument, we know that $a \in \text{span}(\Phi^{-1}(E^+))$ and $a \in \text{span}(\Phi^{-1}(E^-))$. Let E^{++} be the face after E^+ and E^{--} be the face before E^{--}. By condition (iii) of Definition 4.1, a cannot be contained in $\text{span}(\Phi^{-1}(E^{++}))$ or of $\text{span}(\Phi^{-1}(E^{--}))$. But since E^+ and E^{++} are comparable (resp. E^- and E^{--}) we must have $E^+ \subseteq E^{++}$ (resp. $E^{--} \supseteq E^-$). Consider the ordering of the faces we have so far.

$$F^*_{j_1} \supseteq E^+ \subseteq E^{++} \ldots E^{--} \supseteq E^- \subseteq F^*_{j_2}$$

In any case—if $E^{++} = E^{--}$, or if there is some other face between these—we will have some face between $F^*_{j_1}$ and $F^*_{j_2}$ that is not contained in either of its adjacent faces. But this would imply that $F^*_{j_1} = F_i$ and $F^*_{j_2} = F_{i+1}$ are not adjacent faces in $\mathcal{F}$, producing a contradiction. Hence, we must conclude that the assumption was incorrect and $E^+ = E^-$. This proves that the maximum face of F_i is the minimum face of F_{i+1}.

Therefore, we see that $\mathcal{F}$ satisfies all conditions in Definition 3.5 and so it is a cellular string. $\square$

This theorem shows that all pairs in $\mathcal{C}(P)$ are essentially cellular strings on P. It is also true that any given cellular string is represented by a pair (a, ψ) in $\mathcal{C}(P)$. To see this, suppose we are given some f-cellular string $\mathcal{F} = (F_0, \ldots, F_k)$. Then let a be the vector associated to f. For a face F_i in $\mathcal{F}$, let $e_i = \Phi^{-1}(F_i)$ be its corresponding cell of $\mathbb{S}^{n-1}$ and we place a point in the interior of e_i. Since no faces of the cellular string—except possibly F_0 and F_k—are vertices, all e_i will be of dimension $< n-1$. Then, by properties of regular cell complexes, we can connect the point inside e_i to the point inside e_{i+1} by a continuous path contained inside a single $n-1$-cell. The union of these paths gives the path $\varphi_{a,\psi}(t)$ from which we can derive the appropriate ψ.

We can now give $\mathcal{C}(P)$ a cell decomposition according to the cellular string made by each pair. For each cellular string $\mathcal{F}$ on P, we let

$$e_{\mathcal{F}} := \{(a, \psi) \in \mathcal{C}(P) \mid \Delta(a, \psi) \text{ gives cellular string } \mathcal{F}\}$$

be an open cell in this decomposition. The fact that each of these sets is an open cell and that this creates a cell decomposition of $\mathcal{C}(P)$ will be proved in a later paper.

Lemma 4.4. *The set $\mathscr{E} = \{e_{\mathcal{F}}\}$ where $\mathcal{F}$ is a cellular string of P gives a cell decomposition on $\mathcal{C}(P)$. Furthermore, $(\mathcal{C}(P), \mathscr{E})$ is a CW complex.*

Remember in Section 2 when we described the *face poset* of a polytope? That same construction works for a general CW complex. If X is a CW complex, then its face poset, $\mathscr{F}(X)$, is the collection of closed cells in X partially ordered by inclusion.

Consider the face poset of $\mathcal{C}(P)$ with the cell decomposition $\mathscr{E}$. Each element in $\mathscr{F}(\mathcal{C}(P))$ is a cell corresponding with a unique cellular string. We claim that $\overline{e_{\mathcal{F}}} \subseteq \overline{e_{\mathcal{E}}}$ if and only if $\mathcal{E} \subseteq \mathcal{F}$. This claim will be proved in a later paper but follows from the conditions of comparability for cellular strings. If we take it as true for now, then we see that $\mathscr{F}(\mathcal{C}(P))$ is exactly the mega Baues poset.[15] Thus,

[15]The partial order is reversed in this instance, but that does not change the homotopy type of the poset.

the order complex on $\mathscr{F}(\mathcal{C}(P))$ is the same as the order complex on $\Omega(P)$. In order to complete our translation of the mega Baues poset into $\mathcal{C}(P)$, we need to know that the order complex on $\mathscr{F}(\mathcal{C}(P))$ is homotopy equivalent to $\mathcal{C}(P)$.

Theorem 4.5. *The order complex on $\mathscr{F}(\mathcal{C}(P))$ is homotopy equivalent to $\mathcal{C}(P)$.*

This theorem is still awaiting proof. Barring unforeseen circumstances, it will be proved in a later paper.

Corollary 4.6. *$\mathcal{C}(P)$ is homotopy equivalent to $\Omega(P)$.*

4.2 Explicit homotopy to $UT(\mathbb{S}^{n-1})$

We now consider the set of all straight paths in Λ^n.

$$\{(a,c) \in \Lambda^n \mid c : [0,1] \to \mathbb{S}^{n-2} \text{ is a constant map}\}$$

More specifically, this is a subset of $\mathcal{C}(P)$. That is, every straight path gives a cellular string.

Lemma 4.7. *If P is an n-dimensional polytope and $(a,c) \in \Lambda^n$ is a straight path, then $(a,c) \in \mathcal{C}(P)$.*

Proof. We show that (a,c) satisfies all three properties in Definition 4.1 for any polytope P. Let $\mathbb{S}^{n-1}$ have the cell decomposition given by $N(P)$. Also, let $\varphi_{a,c}$ be denoted by φ.

Condition (i). Every cell in $\mathbb{S}^{n-1}$ is convex[16] because it is the intersection of the sphere with a convex cone and the path $\varphi(t)$ is also convex. The shortest path between any two points on the sphere is contained in the great circle through both points. For two points in $\varphi(t)$, this path is traced out by $\varphi(t)$. The intersection of two convex sets is also convex so, for any cell e of $\mathbb{S}^{n-1}$, $e \cap \varphi(t)$ is convex and, in particular, connected. Since φ is a continuous function, we conclude that

$$\varphi^{-1}(e \cap \varphi(t)) = \varphi^{-1}(e)$$

is connected.

Next, we address condition (ii). Let $t_0 \in (0,1)$ such that $\varphi(t_0) \in e$ and $a \notin \operatorname{span}(e)$. Then, because φ is the straight path at $\varphi(t_0)$, we know that the cell before e is the cell below e and the cell after e is the cell above e.

Finally, consider condition (iii). Let $t_0 \in (0,1)$ such that $\varphi(t_0) \in e$ and $a \in \operatorname{span}(e)$. First we note that e must be more than 0-dimensional because no vertices of the cell decomposition (except possibly a and $-a$) contain a in its span. Second, $\varphi(t)$ must intersect e at more than a single point. Otherwise $\operatorname{span}(e)$ would not contain any point of $\varphi(t)$ except $\varphi(t_0)$ since φ is a straight path.[17] Thus, we have a straight line $\varphi(t)$ intersecting a convex cell e of dimension ≥ 1 within the plane of e. This implies that $\varphi(t)$ must cross the boundary of e transversely in two places. Hence, neither of those boundary cells can contain $\varphi(t)$, or by extension a, in their spans. These boundary cells are exactly the cells before and after e. Thus, (a,c) satisfies condition (iii) and we conclude $(a,c) \in \mathcal{C}(P)$. □

[16] In relation to paths on the sphere.

[17] There is some sort of geometry argument here.

Now that we know the set of straight paths is contained in $\mathcal{C}(P)$, let's take a closer look at these straight paths. In choosing a pair (a, c), we pick a vector $a \in \mathbb{S}^{n-1}$ and then pick a point $c \in \mathbb{S}^{n-2}$ which stands for a direction away from a along the surface of the sphere. Hence, the set of straight paths is exactly $UT(\mathbb{S}^{n-1})$. For the last step in this investigation, we need to prove that there is a deformation retraction from $\mathcal{C}(P)$ to the set of straight paths. Our hope is to construct an explicit homotopy for this which will appear in a later paper.

Theorem 4.8. *$\mathcal{C}(P)$ is homotopy equivalent to $UT(\mathbb{S}^{n-1})$.*

Once all the remaining proofs have been completed, Corollary 4.6 and Theorem 4.8 will together provide a proof for the Conjecture 3.12 on page 15.

Acknowledgements

This paper was written as my senior thesis for the 2021-2022 school year. The initial research and background information was studied and written up during Fall 2021 and Winter 2022. Sections 3 and beyond was primarily researched and composed during Spring 2022. I would like to thank the University of Washington math department for providing the opportunity to participate in research as an undergraduate. Also, I would like to acknowledge the Odegaard Library whiteboards for being the medium on which most of this paper took shape. Most of all, I am grateful to my mentor Gaku Liu for inspiring me with an interesting problem, supporting me as I wrestled with complicated ideas, and affirming my crazy choices of names for mathematical objects.

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