(Con)figuring out spaces

Fran Herr

A frustrating scenario

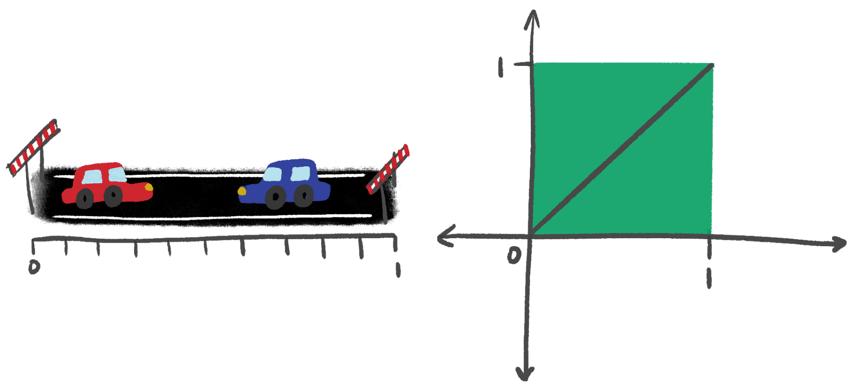
Imagine driving down a narrow road with cars parked on both sides. Suddenly you meet another car driving in the opposite direction. First, you look around for a space to pull over, but you have no luck. So you are forced to back up until you reach the next intersection so that you and the other car can pass each other. Thankfully cars can drive in reverse!

But now we construct a nightmarish hypothetical by placing two cars on a one-lane road with dead ends on both sides. The poor cars can never switch places by driving. We can see this by building another space to model the situation.

The *configuration space* is the set of all possible positions of the two cars on the road—including those reachable if we, acting like a child with toy cars, pick them up and place them down again. We can designate each car's position by a number between 0 and 1: we give '0' to the West end of the road and '1' to the East end. Then each configuration is given by a point in $\mathbb{R}^2$ of the form

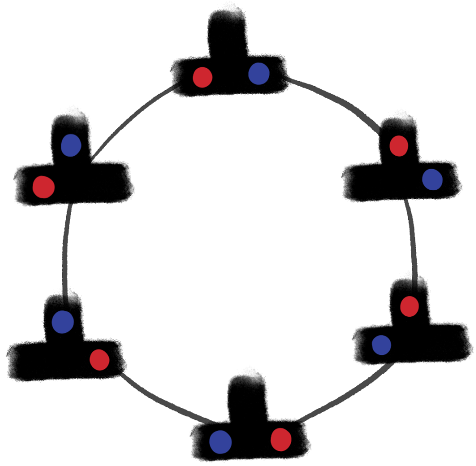
$$(\text{position of car A}, \text{ position of car B}).$$

In total, the configuration space consists of all points in the unit square $[0,1] \times [0,1]$ except for the diagonal (points of the form (x,x)) since the two cars must be in different positions. The fact that the two cars cannot switch their positions (by themselves) is shown by how the configuration space is not connected.



Two cars trapped on a road. The green subset on the right is the configuration space.

Now we mercifully add an alley partway down the road. Then one car can pull aside to let the other pass and thus, all configurations of the two cars on this road can be reached by driving. This means that its configuration space is connected. We can build a model of this configuration space by taking a few of the positions and connecting them by paths if they are definitely reachable by driving.

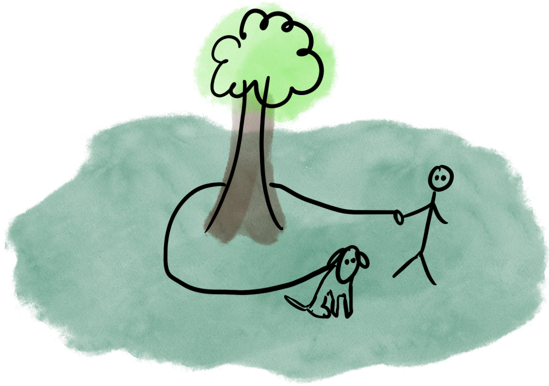


A model of the configuration space of two cars on a road with an alley. The red and blue dots symbolize cars.

Whether a space is connected or not is a very coarse description; there are some more refined questions we can ask. Topologists are interested in getting information about a space by extracting some algebraic structure. One such object we will look at today is the *fundamental group*.

A brief description of the fundamental group

Suppose you are walking a dog on a leash in a field with a tree in the middle. Suddenly Fido bolts off to chase a squirrel, but you stand still in astonishment. Ten seconds later, you call him and he comes running back. Whether or not he made a lap around the tree will make a big difference for you! If not, you can easily pull the leash tight. But if he has, then now you have to go around the tree to untangle the leash. If the squirrel ran around the tree several times and Fido followed him, you will have to make several circles to untangle the leash. For you, the exact path that Fido made does not matter. You only care about the number of times (an in which direction) you now have to walk around the tree.



The fundamental group consists of loops up to wiggling and contracting, which mathematicians call *homotopy.* This wiggling and contracting is like pulling the leash tight after Fido's joy-run. And from our little thought experiment, we see that each loop is characterized by the number of times and the direction it wraps around the tree (we can choose positive numbers for counterclockwise wraps and negative numbers for clockwise wraps). Also, loops add in the same way integers do. If Fido runs once around the tree, twice, it is the same as if he ran two times around the tree. (1 + 1 = 2) If he runs around once around clockwise and once around counterclockwise, he will untangle the leash himself. (1 + -1 = 0)

From this, we can see that the fundamental group of the field with the tree is the integers $\mathbb{Z}$. And likewise, the fundamental group of the configuration space of the second road— the one with the alley— is also $\mathbb{Z}$.

Dancers on stage

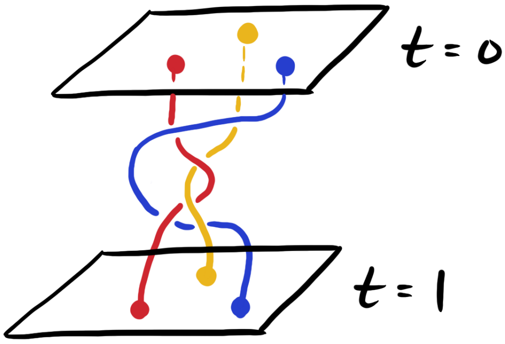
Let's look at a more complicated configuration space. Imagine a group of k dancers on a stage. We will consider the configuration space consisting of all positions of those dancers. We will not be concerned with placement of each dancer's limbs, but will instead only keep track of their spot on the 2-dimensional stage. Certainly the configuration space is connected since the dancers can get from any one configuration to any other. So let's try to find the fundamental group to get more information.

To simplify this situation, we will model each of the k dancers as a point in $\mathbb{R}^2$. Then the configuration space consists of all lists of k distinct points in the plane. It is written as

$$\text{Conf}_k(\mathbb{R}^2) := \{(a_1, \dots, a_k) \mid a_i \in \mathbb{R}^2,\ a_i \neq a_j \text{ if } i \neq j\}.$$

A loop in $\text{Conf}_k(\mathbb{R}^2)$ corresponds to a movement sequence where every dancer returns to their starting position. How can we record such a dance? We could record a video of the dancers, but when we watch it, we will only see one frame at a time. Instead, we can introduce another dimension.

Suppose we have a video of the points moving in the plane which corresponds to the dance. It starts at time 0 and ends at time 1. Then we slice it into many frames and lay them on top of each other with the first frame on the top of the stack and the last frame at the bottom.



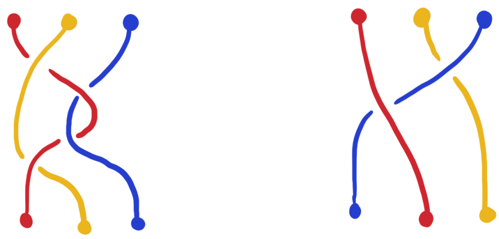
Here is an example if a dance of 3 points. When we stack the “frames” of the video on top of each other, the moving points trace out braiding strands.

If we track one point moving from $t = 0$ to $t = 1$, it will trace out a curve in space. The curves will weave around each other like braiding strings. In this way, each loop in the configuration space can be realized as a braid on k strands.

[insert short-braid.mp4]

The braid group

A braid is a weaving of k strands in three-dimensional space with the top ends pinned in a starting position. There are some braids where the strands return to their starting order at the end of the braid; we call these *perfect braids*.



Two kinds of braids. The left one is a perfect braid, the right one is not.

For a point $(a_1, \ldots a_k)$ in our configuration space, either we care about the order of these k entries, or we don't. In our dance metaphor, this is the difference between dancers who are each in a unique costume and distinguishable vs. dancers who are in uniform costume and indistinguishable.

Mathematically speaking, what we have already defined is the *ordered* configuration space. The *unordered* configuration space $\mathrm{UConf}_k(\mathbb{R}^2)$ is given by $\mathrm{Conf}_k(\mathbb{R}^2)/S_n$ where S_n is the *symmetric group* and it acts by permuting the points. Loops in $\mathrm{Conf}_k(\mathbb{R}^2)$ correspond with perfect braids, where all strands come back to their own spot. And loops in $\text{UConf}_k(\mathbb{R}^2)$ correspond with braids where strands return to any spot.

Then we note that braids do form a group. The identity is given by keeping all strands straight. The inverse of a braid is given by reflecting the braid across the vertical axis. Two braids are added by stacking one on top of another. I will leave you to think about why the collection of pure braids forms a subgroup of all braids.

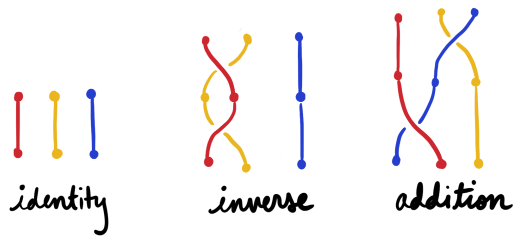


Figure 1: group properties for braids

So we have explained (but not showed) that the fundamental group of $\text{UConf}_k(\mathbb{R}^2)$ is the braid group on k strands, denoted B_k. And the fundamental group of $\text{Conf}_k(\mathbb{R}^2)$ is the *pure braid group* on k strands, denoted P_k.

I love this piece of math because $\text{Conf}_k(\mathbb{R}^2)$ seems like a highly inaccessible space. It contains $2k$ dimensions since there are 2 coordinates specified for the location of each of the k points. However, by changing perspective, we find that its fundamental group is a very understandable and beautiful object. This is one example of how topology can help us access the inaccessible.