Foliating Three-Dimensional Manifolds by Surfaces

Fran Herr
discussed with Danny Calegari

1 Finite Surfaces and Homeomorphisms

1.1 Riemann surfaces

By the classification of surfaces, every connected, closed, orientable surface is either the sphere $\mathbb{S}^2$ or the connected sum of g tori for $g \geq 1$ which we call Σ_g.[1] Throughout this proposal, we will also be working with surfaces with punctures or marked points: $\mathbb{S}^2_k$ and $\Sigma_{g,k}$ will denote a surface with k marked points/punctues.

A topological surface becomes a *Riemann surface* when we give it a *complex structure*. This is an atlas of charts to $\mathbb{C}$ such that the transition maps are holomorphic. All closed orientable surfaces have a complex structure. The sphere $\mathbb{S}^2$ can be identified with $\mathbb{C}P^1$ which is covered by two charts. For Σ_g with $g \geq 1$, represent Σ_g as a $4g$-gon with sides identified. If $g = 1$, we can tile $\mathbb{C}$ with a grid of squares, inducing a complex structure on Σ_1. If $g \geq 2$, we can instead tile $\mathbb{H}^2$ with regular $4g$-gons with $4g$ meeting around a vertex. We realize this with the Poincaré disk model by tiling $\mathbb{D}$ with $4g$-gons with sides made from circular arcs. Since $\mathbb{D} \subset \mathbb{C}$, this induces a complex structure on Σ_g.

Definition 1.1 (Marked Complex Structure)**.** A *marked complex structure* on a smooth surface S is a pair (X, φ) where X is a Riemann surface and $\varphi : S \to X$ is an orientation-preserving diffeomorphism.

Note that φ will also give a local complex structure in charts on S because we can pull back the coordinate charts of X to be charts on S.

Definition 1.2. The *Teichmüller space* $\mathcal{T}(S)$ of a surface S is the set of marked complex structures (X, φ) up to an equivalence relation $\sim$. We say that $(X_1, \varphi_1) \sim (X_2, \varphi_2)$ if there exists a biholomorphism $\psi : X_1 \to X_2$ such that the following diagram commutes up to homotopy.

$$\begin{tikzcd} S \arrow[r, "\varphi_1"] \arrow[dr, "\varphi_2"'] & X_1 \arrow[d, "\psi"] \\ & X_2 \end{tikzcd}$$

Example 1.3. By the Uniformization Theorem, any complex structure on $\mathbb{S}^2$ is conformally equivalent to $\mathbb{C}P^1$. Automorphisms of $\mathbb{C}P^1$ are Möbius transformations given by $f(z) = \frac{az+b}{cz+d}$ for $a, b, c, d \in \mathbb{C}$, $ad - bc \neq 0$. These are all isotopic to Id and so $\mathcal{T}(\mathbb{S}^2)$ is a point.

[1] We will occasionally denote $\mathbb{S}^2$ as Σ_0.

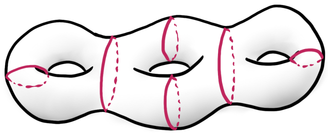


Figure 1: An example of a pants decomposition by 6 curves on Σ_3.

The Uniformization Theorem says that every simply connected Riemann surface is conformally equivalent to $\mathbb{C}$, $\mathbb{D}$, or $\mathbb{C}P^1$. For a genus g Riemann surface $X \cong \Sigma_g$, its universal cover is simply connected and thus conformally equivalent to one of these three options. If $g \geq 2$, then $\widetilde{X}$ is conformally equivalent to $\mathbb{D}$. As a consequence of the Schwarz lemma, the conformal automorphisms of $\mathbb{D}$— given by the group $\mathrm{PSL}(2, \mathbb{R})$— are exactly the isometries of $\mathbb{D}$ with the Poincaré hyperbolic metric. This metric then descends via the covering map $\mathbb{D} \to X$ to give a constant -1 curvature metric on X. Thus, for each complex structure on Σ_g, there is a unique hyperbolic metric up to isometry inducing that complex structure.

Proposition 1.4. *The Teichmüller space $\mathcal{T}(\Sigma_g)$ for $g \geq 2$ is in natural bijection with the set*

$$\{\mu \text{ hyperbolic Riemannian metric on } \Sigma_g\}/\text{isometry}.$$

This other perspective allows us to put natural coordinates– called Fenchel-Neilson Coordinates– on the Teichmüller space corresponding to lengths of certain simple closed curves in Σ_g.

Lemma 1.5 (Pants Decomposition)**.** *There are $3g - 3$ simple closed curves $\{\gamma_i\}$ on Σ_g such that $\Sigma_g \setminus \bigcup_i \gamma_i$ is homeomorphic to the disjoint union of three-holed spheres.*

Once we have decomposed Σ_g into a collection of pairs of pants (three-holed spheres), we can determine a hyperbolic metric on Σ_g by choosing a metric on each pair of pants such that they glue up well.

Lemma 1.6. *A hyperbolic metric on a pair of pants is determined by the lengths of the three boundary components.*

This fact follows by splitting the pair of pants in half to form two hyperbolic hexagons and then applying some basic hyperbolic geometry. The boundaries of the pants will need to glue together by isometries and this requires that matching curves are the same length; this gives $3g-3$ dimensions of freedom. Additionally, when we glue together the curves to form Σ_g, we can do so with any amount of rotation about the γ_i. This adds an additional $3g - 3$ dimensions of freedom.

Proposition 1.7. *The dimension of $\mathcal{T}(\Sigma_g)$ over $\mathbb{R}$ is $6g - 6$.*

A theme of this proposal— and of all my mathematical interests— is the relationship between a geometric structure on a manifold and its symmetry group. In this case, we will study symmetries of hyperbolic metrics on Σ_g.

Definition 1.8 (hyperelliptic surface 1)**.** A Riemann surface $X \cong \Sigma_g$, $g \geq 2$, is *hyperelliptic* if there is an involution $\iota : X \to X$ (i.e. an order 2 conformal automorphism) with $2g + 2$ fixed points.

We can perform an Euler characteristic computation using the theory of orbifold Euler characteristic to find that $\chi(X/\iota) = 2$. Since ι was orientation preserving, X/ι is connected, closed, and orientable. Hence, it must be the sphere $\mathbb{S}^2$. This gives an equivalent definition of hyperelliptic surfaces.

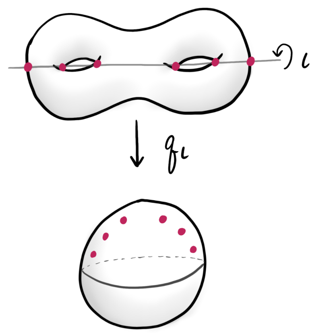


Figure 2: A hyperelliptic involution ι and its corresponding branched covering q_ι on Σ_2.

Definition 1.9 (hyperelliptic surface 2)**.** A Riemann surface $X \cong \Sigma_g$ is *hyperelliptic* if it has a two-sheeted holomorphic branched cover to $\mathbb{S}^2$ with $2g+2$ branch points.

Let S be a surface with marked points $p_1, \ldots, p_n$. A complex orbifold structure on S is given by a complex structure on $S \setminus \{p_1, \ldots, p_n\}$ such that at a marked point p_i there is a neighborhood which is conformally equivalent to the image of $\mathbb{D}$ under the map $z \mapsto z^k$ for some $k \geq 2$. The value k is the *order of the cone point*.

The two-sheeted holomorphic branched cover $X \to X/\iota \cong \mathbb{S}^2$ gives a correspondence between complex structures on X which are symmetric with respect to ι and complex orbifold structures on $\mathbb{S}^2$ with $2g+2$ order 2 cone points.

$$\left\{ \begin{array}{c} \text{hyperelliptic Riemann surfaces} \\ X \cong \Sigma_g \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{complex structures on } \mathbb{S}^2_{2g+2} \\ \text{with order 2 cone points} \end{array} \right\}$$

Thus, the space of hyperelliptic Riemann surfaces of genus g is parameterized by $\mathcal{T}(\mathbb{S}^2_{2g+2})$; this gives a natural embedding of $\mathcal{T}(\mathbb{S}^2_{2g+2})$ into $\mathcal{T}(\Sigma_g)$. The placement of the marked points in $\mathbb{C}P^1$ induces such a complex orbifold structure on $\mathbb{S}^2$. Recall that $\mathrm{Aut}(\mathbb{C}P^1) \cong$ Möb $\cong \mathrm{PSL}(2, \mathbb{C})$ and this group acts transitively on triples of points in $\mathbb{C}P^1$. Then the orbifold structure is determined by the placement of $(2g+2) - 3 = 2g-1$ points in $\mathbb{C}P^1$. This gives $\dim_{\mathbb{R}} \mathcal{T}(\mathbb{S}^2_{2g+2}) = 4g-2$ which is also the dimension of the space of hyperelliptic surfaces of genus g.

1.2 Mapping class groups

The *mapping class group* of a surface is the group of orientation preserving homeomorphisms up to homotopy.

$$\mathrm{MCG}(\Sigma_{g,k}) = \mathrm{Homeo}^+(\Sigma_{g,k})/\sim$$

The equivalence classes of homeomorphisms in $\mathrm{MCG}(\Sigma_{g,k})$ are called *mapping classes*. Multiplication of two mapping classes is given by the composition of representatives for each class, and this is independent of choice. That is, if $f_1 \sim f_2$ and $g_1 \sim g_2$, then $f_1 \circ g_1 \sim f_2 \circ g_2$. This shows $\mathrm{MCG}(\Sigma_{g,k})$ is a well-defined group.

Remark 1.10. Every homeomorphism on $\Sigma_{g,k}$ is isotopic to a diffeomorphism [1]. We can then equivalently define $\mathrm{MCG}(S) = \mathrm{Diffeo}^+(S)/\sim$ and always take a smooth representative.

One very important example of a mapping class is a *Dehn twist.* For a simple closed curve $\gamma \in \Sigma_{g,k}$ we define $[T_\gamma] \in \mathrm{MCG}(\Sigma_{g,k})$. Let T_γ be the identity outside an annular neighborhood A around γ. Give A a coordinate function to a complex annulus.

$$\phi : A \to \left\{ re^{i\theta} \in \mathbb{C} \,\middle|\, 1 < r < 2,\ 0 \leq \theta < 2\pi \right\}$$

Then define a map *twist* on this complex annulus by

$$\mathit{twist} : re^{i\theta} \mapsto re^{i\theta} e^{i(1-r)\pi}.$$

And we define $T_\gamma|_A = \phi^{-1} \circ \mathit{twist} \circ \phi$. Note that *twist* is the identity on the boundary of the annulus so T_γ is continuous. See Figure 3 for the action of *twist*.

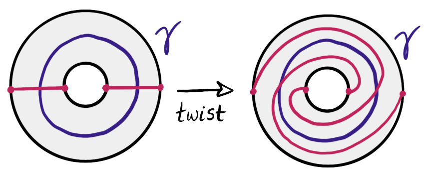


Figure 3: The action of *twist* on the complex annulus.

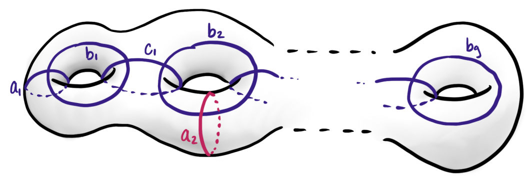


Figure 4: A set of $2g+1$ simple closed curves on Σ_g. Dehn twists about these curves generate $\mathrm{MCG}(\Sigma_g)$.

Proposition 1.11 (Dehn-Lickorish, Humphries [1], statement only)**.** *The mapping class group* $\mathrm{MCG}(\Sigma_g)$ *for* Σ_g*,* $g \geq 2$*, is finitely generated by* $2g+1$ *Dehn twists about simple closed curves.*

The mapping class group acts on Teichmüller space. If $\phi : S \to S$ is a diffeomorphism, and (X, φ) is a complex structure on S, then $(X, \varphi \circ \phi^{-1})$ is another marked complex structure on S. This resulting complex structure is independent of the choice of diffeomorphism up to homotopy: if we take ϕ and ϕ' to be homotopic diffeomorphisms, then $\varphi \circ \phi$ and $\varphi \circ \phi'$ will also be homotopic. This gives a well-defined action $\mathrm{MCG}(\Sigma_{g,k}) \circlearrowright \mathcal{T}(\Sigma_{g,k})$.

Remark 1.12. The quotient $\mathcal{T}(\Sigma_{g,k}) / \mathrm{MCG}(\Sigma_{g,k})$ is called the *moduli space* of the surface denoted $\mathcal{M}_{g,k}$. This an orbifold of the same dimension as $\mathcal{T}(\Sigma_{g,k})$. The cone points of $\mathcal{M}_{g,k}$ correspond to Riemann surfaces with extra symmetry. Moduli spaces do play a key role in the larger story, but it is not a focus for this paper.

Remark 1.13. So far, we have been discussing the mapping class group for general surfaces $\Sigma_{g,k}$. Moving forward, we will focus specifically on closed surfaces Σ_g but a version of the following material could be applied to surfaces with punctures.

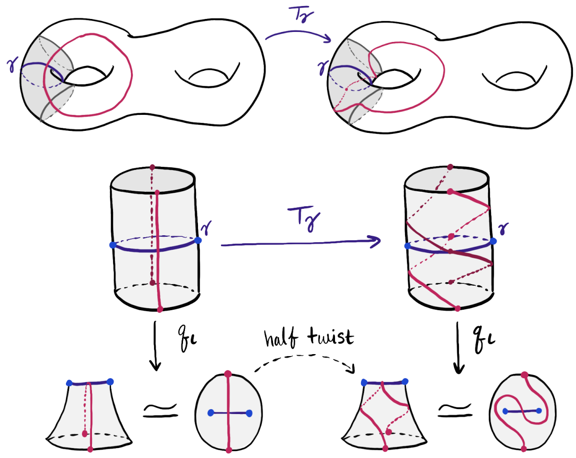


Figure 5: If γ is a symmetric curve with respect to ι, then T_γ is hyperelliptic. By restricting to an annular neighborhood of γ and looking at the diagram, we find that T_γ descends to a half twist on the quotient space (a disk with 2 marked points).

A *hyperelliptic involution* on a topological surface Σ_g is like the map ι referenced in Definition 1.8 but without necessarily preserving any conformal structure. We just require $\iota : \Sigma_g \to \Sigma_g$ to be an orientation-preserving homeomorphism and satisfy $\iota^2 = \mathrm{Id}$ with $2g + 2$ fixed points.

Lemma 1.14. *If ι_1 and ι_2 are two hyperelliptic involutions on Σ_g, then $[\iota_1]$ and $[\iota_2]$ are conjugate in* $\mathrm{MCG}(\Sigma_g)$.

In light of Lemma 1.14, we will fix one hyperelliptic involution on Σ_g and call it ι. The following constructions are for this fixed ι but are conjugate to those for any other hyperelliptic involution.

Definition 1.15. A mapping class $[f] \in \mathrm{MCG}(\Sigma_g)$ is *hyperelliptic* if there is a representative f satisfying the equivalent conditions:

1. f commutes with ι up to homotopy;
2. f descends to a homeomorphism on Σ_g/ι.

Note that [Id] satisfies this definition and that if $[f]$ is hyperelliptic, then $[f^{-1}]$ is as well. Also, the composition of two hyperelliptic mapping classes is also hyperelliptic. Thus, the collection of all these mapping classes form a subgroup of $\mathrm{MCG}(\Sigma_g)$.

Definition 1.16. The *hyperelliptic (or symmetric) mapping class group* $\mathrm{HMCG}(\Sigma_g)$ is the subgroup of $\mathrm{MCG}(\Sigma_g)$ containing all the hyperelliptic mapping classes.

Example 1.17. Suppose that γ is a simple closed curve in Σ_g such that $\iota(\gamma) = \gamma$. Then T_γ commutes with ι so $[T_\gamma]$ is hyperelliptic. This can be seen by restricting to an annular neighborhood of γ. See Figure 5.

Proposition 1.18. *The fixed set under the action of* $[\iota]$ *on* $\mathcal{T}(\Sigma_g)$ *is the set of all hyperelliptic surfaces for which* ι *is a conformal automorphism; we denote this* $\mathrm{hyp}(\Sigma_g) \subset \mathcal{T}(\Sigma_g)$. *Further, the action by* $\mathrm{HMCG}(\Sigma_g)$ *fixes* $\mathrm{hyp}(\Sigma_g)$ *as a set.*

This proposition links the geometry in Section 1.1 with an algebraic structure, namely the mapping class group. Following Example 1.17, if a simple closed curve γ has a symmetric representative, then T_γ is hyperelliptic. Consider the generators for $\mathrm{MCG}(\Sigma_g)$ in Figure 4. When $g = 2$, all the generators have a symmetric representative. This means that for genus 2,

$$\mathrm{HMCG}(\Sigma_2) = \mathrm{MCG}(\Sigma_2).$$

However, when $g \geq 3$, the curve a_2 is the only generator with no symmetric representative. So for $g \geq 3$, $\mathrm{HMCG}(\Sigma_g)$ is strictly smaller than $\mathrm{MCG}(\Sigma_g)$.

2 Three-Manifolds

We now transition to studying manifolds of three dimensions. In Sections 2.2 and 3, we give a three-manifold M extra structure by writing it as the disjoint union of surfaces– a *foliation*. We can use the geometry of surfaces and their mapping class groups developed in the first section to help us understand M.

If M and N are oriented 3-manifolds, their *connected sum* $M\#N$ is formed by removing a 3-ball from each and gluing along the boundary sphere with an orientation reversing homeomorphism. Note that if M and N are not homeomorphic to S^3, then the gluing sphere is essential in $M\#N$. A 3-manifold M is *prime* if whenever $M = A\#B$, either $A \cong S^3$ or $B \cong S^3$.

The Prime Decomposition Theorem of Kneser and Milnor says that every compact 3-manifold M can be written as $M = M_1\#M_2\# \ldots \#M_n$ where each M_i is prime. This is unique up to homeomorphism of the M_i and permutation of the pieces.

Definition 2.1 (Irreducible)**.** A 3-manifold M is *irreducible* if every embedded $\mathbb{S}^2 \hookrightarrow M$ bounds an embedded ball.

Notice that if M is irreducible, then M is also prime. On the other hand, $M = \mathbb{S}^2 \times S^1$ is the only prime manifold which is not irreducible. Moving forward, we will often require that M is irreducible. This is because the theorems we state are often simple in the case $M = \mathbb{S}^2 \times S^1$. Then, once we understand all prime 3-manifolds, we can use prime decomposition to understand all 3-manifolds.

2.1 Thurston norm

Suppose that M is an irreducible, oriented, compact 3-manifold with incompressible boundary (i.e. $\pi_1(\partial M) \to \pi_1(M)$ is an injection). Let $S \hookrightarrow M$ be a two-sided properly embedded surface and let $\{S_i\}$ be the set of connected components of S which are not homeomorphic to $\mathbb{S}^2$ or D^2. We define

$$\chi^-(S) := \sum_i \chi(S_i)$$

Note that $[S] = [\cup_i S_i]$ in $H_2(M, \partial M; \mathbb{Z})$; this follows because M is irreducible with incompressible boundary so every embedded sphere is contractible and thus 0 in homology.

Lemma 2.2 (Embedded Surface [2])**.** *Let M be a compact two-sided 3-manifold. Every class $\alpha \in H_2(M, \partial M; \mathbb{Z})$ is represented by $[S]$ where S is an oriented compact surface properly embedded in M.*

Lemma 2.2 allows us to define a map $\|\cdot\| : H_2(M, \partial M; \mathbb{Z}) \to \mathbb{R}$ given by

$$\|\alpha\| = \min\{-\chi^-(S) \mid [S] = \alpha\}.$$

This is the first step in defining the *Thurston norm* which is a norm on the vector space $H_2(M, \partial M; \mathbb{R})$.

Proposition 2.3 (Semi-norm)**.** *The map defined above on $H_2(M, \partial M; \mathbb{Z})$ is a* semi-norm. *That is, it satisfies the following properties.*

1. $\|\alpha\| \geq 0$ *for all* $\alpha \in H_2(M, \partial M; \mathbb{Z})$
2. $\|k\alpha\| = |k|\|\alpha\|$ *for all* $k \in \mathbb{Z}$
3. $\|\alpha + \beta\| \leq \|\alpha\| + \|\beta\|$ *for all* $\alpha, \beta \in H_2(M, \partial M; \mathbb{Z})$

Proof sketch. –

1. By definition, $\chi^-(S) \leq 0$ for all S since all spherical and disk components have been excluded. Thus, $\|\alpha\| \geq 0$.
2. Let S be a surface with $\|\alpha\| = -\chi^-(S)$. We can represent $k\alpha$ by $|k|$ parallel copies of S. This gives $\|k\alpha\| \leq |k|\|\alpha\|$. For the other inequality, we can always find a smooth map $f : M \to S^1$ such that $f^{-1}(0) = S$. Then we can lift f to the $|k|$-fold cover of S^1 by the lifting criteria. If $\hat{f}$ is the lifted map, this implies that $\hat{f}^{-1}(0)$ consists of $|k|$ disjoint copies of S. Thus, every surface representing $k\alpha$ must be in this form and $\|k\alpha\| = |k|\|\alpha\|$.
3. We represent α and β with Thurston-norm minimizing surfaces S and T. We can assume these are incompressible surfaces which intersect in minimal position. Using a cut and paste argument while preserving Euler characteristic, we can modify $S \cup T$ to obtain a properly embedded surface representing $[\alpha + \beta]$. This gives an upper bound on $\|\alpha + \beta\|$.

□

A manifold M is *atoroidal* if there are no essential tori in M (i.e. no embedded non-boundary parallel incompressible tori). A manifold with boundary is *acylindrical* if its double is atoroidal.

Proposition 2.4. *If M is atoroidal (and acylindrical if $\partial M \neq \emptyset$), then $\|\cdot\|$ is positive definite so it is a norm. That is, $\|\alpha\| = 0$ if and only if $\alpha = [0]$ in homology.*

Proof for M without boundary. Suppose that $\|\alpha\| = 0$ and α is represented by an incompressible surface S with $\|\alpha\| = -\chi^-(S)$. Let $\{S_i\}$ be the non-spherical connected components of S so that $\chi^-(S) = \sum_i \chi(S_i) = 0$. Note that $\chi(S_i) \leq 0$ for each component S_i since S_i is not $\mathbb{S}^2$. Hence, we conclude that $\chi(S_i) = 0$ for all components S_i. Since S is orientable, all components must be tori and S is the disjoint union of tori. However, M is atoroidal so S must be compressible. Thus, $\alpha = [0]$. □

We extend $\|\cdot\|$ to $H_2(M, \partial M; \mathbb{R})$ by first extending to homology classes with rational coefficients by using Proposition 2.3 part (2). Then we use the convexity in Proposition 2.3 part (3) and limiting sequences to define $\|\alpha\|$ for all $\alpha \in H_2(M, \partial M; \mathbb{R})$. By Lefschetz Duality, $H_2(M, \partial M; \mathbb{R}) \cong H^1(M; \mathbb{R})$ so we can think of $\|\cdot\|$ as a norm on first cohomology. Then we can associate $\varphi \in H^1(M; \mathbb{R})$ with a 1-form μ in $H^1_{dR}(M; \mathbb{R})$ such that $\varphi(\gamma) = \int_\gamma \mu$ for all closed curves γ. This μ will always be a closed form because $\varphi(\gamma) = \varphi(\gamma')$ for $[\gamma] = [\gamma']$ in $H_1(M; \mathbb{R})$.

Proposition 2.5. *Let $\varphi \in H^1(M;\mathbb{R})$ and μ be the 1-form such that $\varphi(\gamma) = \int_\gamma \mu$ for all closed curves γ. Then μ is non-singular if and only if M fibers over the circle with fiber S such that $\|\varphi\| = -\chi(S)$.*

Proof. Suppose M fibers over the circle with fibration $f : M \to S^1$ and fiber S which is Thurston-norm minimizing in its homology class. Then we can take $\mu = f^* d\theta$ and this is the desired 1-form.

On the other hand, suppose there is such a 1-form μ with $\varphi(\gamma) = \int_\gamma \mu$. Let $\{\gamma_i\}$ be the (finitely many) generators of $\pi_1(M)$. Then we perturb μ while keeping it non-singular so that $\int_{\gamma_i} \mu \in \mathbb{Q}$ for all generators γ_i. Let N be a large integer so that $\int_{\gamma_i} \mu = \frac{k_i}{N}$ for all γ_i. Then $\mu' = N\mu$ is also non-singular and $\int_{\gamma_i} \mu' \in \mathbb{Z}$ for all generators γ_i. Then we fix some $x_0 \in M$ and define a map $f : M \to \mathbb{R}/\mathbb{Z}$ by

$$f(x) = \left(\int_{x_0}^{x} \mu' \right) / \mathbb{Z}.$$

This map is well-defined since $f(x)$ is independent of path because $\int_\gamma \mu' \in \mathbb{Z}$ for all closed curves γ. We can see that f is indeed a fibration by applying Ehresmann's theorem since f is a submersion and M is compact. □

Now observe that $H_2(M, \partial M; \mathbb{R}) \cong \mathbb{R}^{b_2}$ where b_2 is the second betti number. Then we have a lattice of integral homology classes $\mathbb{Z}^{b_2} \subset H_2(M, \partial M; \mathbb{R})$. Note that $\|\cdot\|$ is an integer on all these lattice points. This allows us to use the following fact: if a norm on $\mathbb{R}^d$ takes on integer values on $\mathbb{Z}^d$, then the unit ball is a finite-sided polyhedron.

Theorem 2.6 (Thurston [3])**.** *The unit ball $B_{\|\cdot\|}$ in $H_2(M, \partial M; \mathbb{R})$ with respect to the Thurston norm $\|\cdot\|$ is a finite-sided polyhedron.*

Proposition 2.7 (statement only)**.** *The set $\mathcal{C}$ of non-singular closed 1-forms is the union of cones over faces of $B_{\|\cdot\|}$ in $H^1_{dR}(M;\mathbb{R})$.*

Example 2.8 (Whitehead Link). Let W be the whitehead link with link components ℓ_1 and ℓ_2 and let $M = S^3 \setminus W$. Then we note that

$$H_2(M, \partial M; \mathbb{Z}) \cong H^1(M; \mathbb{Z}) \cong \operatorname{Hom}\left(H_1(M;\mathbb{Z}), \mathbb{Z}\right).$$

Note that $H_1(M;\mathbb{Z})$ is generated by the homology classes of two curves γ_1 and γ_2 linking with ℓ_1 and ℓ_2 respectively. So $H_1(M;\mathbb{Z}) \cong \mathbb{Z}^2$ and $\operatorname{Hom}(H_1(M;\mathbb{Z}), \mathbb{Z})$ is determined by where it sends these generators. This gives $H_2(M, \partial M; \mathbb{Z}) \cong \mathbb{Z}^2$ and it is generated by α_1 and α_2 where α_i is the class of surfaces with ℓ_i as a boundary.

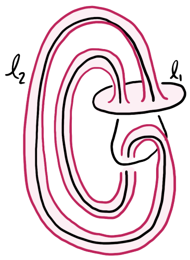
We can find explicitly a surface $S_i \cong T^2 \setminus \{\text{disk}\}$ with ℓ_i as a boundary as shown in Figures 6a and 6b. We calculate

$$\|\alpha_i\| \le -\chi(S_i) = 1.$$

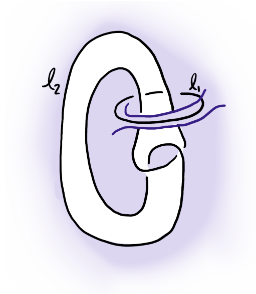
There is no orientable surface with one boundary component and Euler characteristic 0; this means $\|\alpha_i\| \neq 0$. Hence, α_1 and α_2 both lie on the unit ball $B_{\|\cdot\|}$. Next consider the class $\alpha_1 + \alpha_2$. By convexity, we know

$$\|\alpha_1 + \alpha_2\| \le \|\alpha_1\| + \|\alpha_2\| = 2.$$

A surface with two boundary components has an even Euler characteristic. So either $\|\alpha_1 + \alpha_2\| = 0$ and there is an annulus with $\ell_1 \cup \ell_2$ as a boundary, or $\|\alpha_1 + \alpha_2\| = 2$. This first case cannot happen. By applying this same argument to $\alpha_1 - \alpha_2$ and then scaling by $\frac{1}{2}$ we obtain four more points on $B_{\|\cdot\|}$. Since the unit ball is a finite sided convex polytope, we have determined it to be the square with vertices $\pm(1,0)$ and $\pm(0,1)$.



(a)



(b)

Figure 6: The complement of the Whitehead link with two embedded surfaces with minimal genus in their homology class. Both surfaces are $T^2 \setminus \{\text{disk}\}$.

2.2 3-Manifolds fibering over S^1

In this section, we will examine a rich family of 3-manifolds called *mapping tori*. A mapping torus is constructed by a homeomorphism f on a surface. We will see how the algebraic properties of f in $\mathrm{MCG}(S)$ give specific geometric structure to its mapping torus.

Definition 2.9 (Mapping torus)**.** Let $f : S \to S$ be an orientation preserving homeomorphism of an oriented surface S. The *mapping torus* M_f of f is

$$M_f := S \times [0,1] \Big/ (x,1) \sim (f(x),0).$$

A mapping torus M_f naturally fibers over the circle with the map $F(x,t) = t$ and fiber $F^{-1}(t) \cong S$. Not only do mapping tori fiber over the circle, but the converse is also true. If M is a fiber bundle over the circle, it must be a mapping torus. Let the fibering $F : M \to S^1$ have a monodromy given by a homeomorphism $f : F^{-1}(0) \to F^{-1}(0)$. By definition, $M \cong M_f$.

Mapping tori provide a nice invariant for members of the mapping class group of a finite surface Σ_g. If $[f] = [h]$ in $\mathrm{MCG}(\Sigma_g)$, then $M_f \cong M_h$ with a homeomorphism preserving the fibration. In fact, this is also true if $[f]$ and $[h]$ are conjugate in $\mathrm{MCG}(\Sigma_g)$. However, there do exist non-conjugate mapping classes $[f]$ and $[h]$ such that $M_f \cong M_h$, but the homeomorphism will not preserve the fibration.

By looking a little closer at the different types of homeomorphisms on Σ_g, we can learn more about these mapping tori.

Theorem 2.10 (Neilson-Thurston Classification, statement only)**.** *Let $f : \Sigma_g \to \Sigma_g$ be a homeomorphism. Then f is homotopic to some homeomorphism g such that either*

1. *g is* finite order*: there is some $k \geq 0$ such that $g^k = \mathrm{Id}$;*
2. *g is* reducible*: there is a nonempty collection of isotopy classes of disjoint essential simple closed curves $\{c_i\}$ such that $\{g(c_i)\} = \{c_i\}$;*
3. *g is* pseudo-Anosov*: there is a pair of transverse measured singular foliations $(\mathcal{F}^u, \mu_u)$ and $(\mathcal{F}^s, \mu_s)$ and $\lambda > 1$ such that*

$$g \cdot (\mathcal{F}^u, \mu_u) = (\mathcal{F}^u, \lambda\mu_u) \quad \textit{and} \quad g \cdot (\mathcal{F}^s, \mu_s) = (\mathcal{F}^s, \lambda^{-1}\mu_s).$$

Examples of homeomorphisms of type (1) and (2) in Theorem 2.10 are readily available. The hyperelliptic involution in Figure 2 has order 2. To construct a reducible homeomorphism, let $\gamma_1, \ldots \gamma_k$ be disjoint simple closed curves in Σ_g. Then the composition of Dehn twists $T_{\gamma_1} \circ \cdots \circ T_{\gamma_k}$ is reducible because each γ_i is fixed by this homeomorphism.

Pseudo-Anosov homeomorphisms are more elusive. First, we consider *Anosov homeomorphisms* on the torus. Such a map $g : T^2 \to T^2$ is characterized by the existence of a pair of transverse measured (non-singular) foliations $(\mathcal{F}^u, \mu_u)$ and $(\mathcal{F}^s, \mu_s)$ and $\lambda > 0$ satisfying the same criteria as in (3) of Theorem 2.10. Any matrix $A \in \mathrm{SL}(2, \mathbb{Z})$ with $|\mathrm{tr} A| > 2$ will satisfy this condition; the two sets of lines parallel to the eigenspaces will be the foliations. Then we construct a branched covering $\Sigma_g \to T^2$ branched over fixed points of A. We can lift the Anosov map on T^2 to a pseudo-Anosov map on Σ_g. The singular foliations will be the lift of the foliations on T^2.

Theorem 2.11 (Thurston, statement only)**.** *Let $[f] \in \mathrm{MCG}(\Sigma_g)$ with $g \geq 2$. Let M_f be the mapping torus. Then,*

1. *if f is finite order, M_f is finitely covered by $\Sigma_g \times S^1$ and has geometry $\mathbb{H}^2 \times \mathbb{E}$;*
2. *if f is reducible then M_f contains an embedded essential torus. We can cut along this torus to decompose M_f into a mapping tori of a surface with boundary;*
3. *f is pseudo-Anosov if and only if M_f is hyperbolic.*

Theorem 2.10 and 2.11 together allow us to always put a geometry on M_f so that the fibers $\Sigma_g \times \{t\}$ are hyperbolic. In the case where f is pseudo-Anosov, we can find a hyperbolic metric on the whole manifold M_f.

3 Taut Foliations

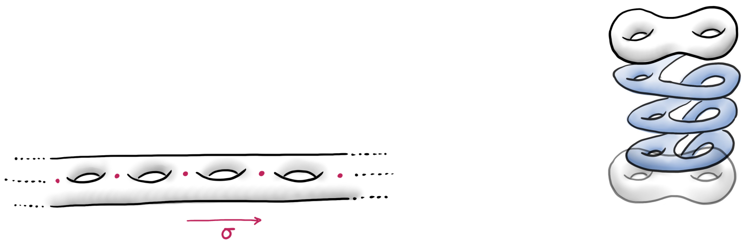
Definition 3.1 (taut foliation)**.** A co-oriented foliation $\mathcal{F}$ of a 3-dimensional M by surfaces is *taut* if there is a closed oriented transversal intersecting every leaf of $\mathcal{F}$.

Example 3.2. The foliation $\mathcal{F}$ of a mapping torus M_f comprised of leaves $\Sigma_g \times \{t\}$ is a taut foliation. Take the path $x \times [0, 1]$ for some $x \in \Sigma_g$. Then add in a path between the endpoints $(x, 0)$ and $(f(x), 0)$ lying in the leaf $\Sigma_g \times \{0\}$ to obtain a closed curve γ. We can perform a homotopy on γ to make it transverse to every leaf of $\mathcal{F}$. Since f is an orientation-preserving homeomorphism, we can assign a co-orientation to this transversal.

There are many different equivalent formulations of a taut foliation; each is useful in a different context. We list a few which are most relevant to us but a more extensive list can be found in [2]. Number (6) in Proposition 3.3 will be of primary importance.

Proposition 3.3 (Equivalent definitions of tautness)**.** *The following are equivalent conditions for $\mathcal{F}$ on M^3 being a taut foliation.*

1. *For every point $p \in M$, there is an immersed circle transverse to $\mathcal{F}$ passing through p.*
2. *There is no proper closed submanifold N of M whose boundary is tangent to $\mathcal{F}$ and for which the co-orientation points in to N along ∂N.*
3. *There is a closed 2-form ω on M positive on $T\mathcal{F}$.*
4. *There is a flow X transverse to $\mathcal{F}$ which is volume preserving for some Riemannian metric on M.*



(a) The shift map σ on S. The pink points represent one orbit under iteration by σ.

(b) A heuristic picture of M_σ for the shift σ on the ladder surface which gives an example of a depth 1 foliation.

Figure 7

5. *There is a Riemannian metric on M for which every leaf of $\mathcal{F}$ is a minimal surface.*

6. *There is a map $f : M \to \mathbb{S}^2$ whose restriction to each leaf λ is a branched covering.*

Definition (6) is the main formulation that we will use in this paper. There are two analytic proofs that this is equivalent to Definition 3.1: one generalizing Poincaré's classical construction of meromorphic functions on Riemann surfaces and the other follows a route with Eliashberg-Thurston, contact stuructures, and symplectic 4-manifolds, applying Donaldson's Theorem. In [4], Calegari gives a two combinatorial proofs. One of these constructs Belyi maps from a dual to a Voroni tesselation.

Proposition 3.4. *An embedded, oriented, closed surface $S \hookrightarrow M$ is Thurston norm minimizing if and only if it is the compact leaf of a taut foliation on M.*

The proof of the backwards direction of this proposition (compact leaf of taut foliation $\Rightarrow$ Thurston norm minimizing) uses the Euler class of the foliation $e(\mathcal{F})$. It follows from the inequality $|e(\mathcal{F})[S]| \leq |\chi^-(S)|$ for all embedded, oriented, closed surfaces S and the fact that equality is achieved when S is a compact leaf of $\mathcal{F}$.

The forward direction (Thurston norm minimizing $\Rightarrow$ compact leaf of a taut foliation) is a theorem of Gabai [5]. The proof proceeds by decomposing M along Thurston norm minimizing surfaces to obtain taut sutured manifold pieces. Since taut sutured manifolds have finite hierarchies, this procedure will terminate. Then the pieces of the decomposition can be foliated with non-compact leaves and the decomposing surfaces are compact leaves of the resulting foliation. This extensive argument involves developing the theory of sutured manifolds and I will not give the proof in my exam.

3.1 Finite depth foliations and big mapping tori

Definition 3.5. Compact leaves of a foliation are are *depth 0*. A leaf λ is *depth k* if $\overline{\lambda}$ is a union of depth $k-1$ leaves. A foliation is *depth k* if all leaves are depth $\leq k$.

In this proposal, we will only be discussing depth 0 and depth 1 foliations. For a picture for a depth 1 foliation, one should imagine a $\mathbb{Z}$ cover of a finite surface Σ_g spiraling around and converging to Σ_g. A schematic is shown in Figure 7b. To construct a depth 1 foliation explicitly, we can take a homeomorphism on an *infinite-type surface* and build its corresponding mapping torus.

Example 3.6 (A shift on a ladder). Let S be a *ladder surface*. This can be defined as a cylinder $S^1 \times \mathbb{R}$ with a handle attached from $S^1 \times \{n\}$ to $S^1 \times \{n+1\}$ for all even integers n. Or define S as a $\mathbb{Z}$-cover of Σ_2. See Figure 7a.

Then we define a *shift map* $\sigma : S \to S$ as a homeomorphism which shifts the handle based at n to the handle based at $n+2$. In Figure 7a, the pink points represents one orbit under σ. The shift map σ has one *repelling end* on the left of S and one *attracting end* on the right. We form the mapping torus in the same way as with finite-type surfaces.

$$M_\sigma := S \times [0,1] \Big/ (x,1) \sim (\sigma(x), 0)$$

Then M_σ is a non-compact 3-manifold foliated by $S \times \{t\}$. We give it a natural compactification $\overline{M_\sigma}$.

Consider the set of all escaping rays γ in M_σ. We will call two rays γ_1 and γ_2 equivalent if they agree outside of a compact subset of M_σ. For each equivalence class of escaping rays, add in a point of $\partial\overline{M_\sigma}$. In the result, we add on two genus 2 surfaces— one for the repelling end and one for the attracting end. See Figure 7b. The foliation $\mathcal{F}$ of $\overline{M_\sigma}$ consists of all non-compact leaves homeomorphic to S and the two compact Σ_2 leaves; $\mathcal{F}$ is an example of a depth 1 foliation. The compact boundary leaves are Thurston norm-minimizing by Proposition 3.4.

Remark 3.7. Above we describe the shift map by one genus. We could also define σ_k on the ladder surface analogously as a shift map by k genera.

We now generalize the construction in Example 3.6.

Definition 3.8. A surface S is of *infinite-type* if its fundamental group is infinitely generated. An *end* of an infinite-type surface is an equivalence class of *exiting sequences*: a sequence of connected open sets $\{U_n\}_{n\in\mathbb{N}}$ satisfying

1. $U_n \subset U_m$ if $n > m$;
2. $\overline{U_n}$ is not compact for any n;
3. U_n has compact boundary for all n;
4. any compact subset K of S is contained in finitely many U_n's.

Two exiting sequences $\{U_n\}$ and $\{V_n\}$ are equivalent if there are $i, j \in \mathbb{N}$ and $i', j' \in \mathbb{N}$ such that $U_i \subset V_j$ and $U_{i'} \supset V_{j'}$. An end is *accumulated by genus* if it has no exiting sequence $\{U_i\}$ where U_i is eventually homeomorphic to a cylinder for large i.

Definition 3.9. A homeomorphism f on an infinite-type surface S is *end-periodic* if outside of a compact subset K of S, f acts by a shift map.

Definition 3.10. Let f be an end-periodic homeomorphism on an infinite-type surface S. The *compactified mapping torus* $\overline{M}_f$ is M_f as in Example 3.6 with a boundary component E/f for each end E. The natural foliation $\mathcal{F}$ on $\overline{M}_f$ is given by the collection of non-compact leaves $S \times \{t\}$ along with the compact leaves E/f.

Proposition 3.11 (statement only). *Let M be a compact 3-manifold with a depth 1 foliation $\mathcal{F}$. Then M can be dissected into pieces along compact leaves of $\mathcal{F}$ so that each piece is foliated as $\overline{M}_f$ so some end-periodic homeomorphism on an infinite-type surface.*

Proposition 3.11 reveals why we focus on compactified mapping tori of infinite type surfaces: these are essentially all depth 1 foliations. Very little in known about the group of end-periodic homeomorphisms on infinite-type surfaces. We hope the connection between algebraic properties of an end-periodic $[f] \in \mathrm{MCG}(S)$ and the geometric properties of the foliated 3-manifold $\overline{M}_f$ will help us learn more about both objects. In particular, following Proposition 3.3 part (6), we are interested in the minimum complexity of the leafwise branched covering map on $\overline{M}_f$ and whether this reflects some measure of complexity of an end-periodic homeomorphism f.

References

[1] B. Farb and D. Margalit, *A Primer on Mapping Class Groups.* Princeton University Press, 2012.

[2] D. Calegari, *3-manifolds.* in progress.

[3] W. P. Thurston, "A norm for the homology of 3-manifolds," *Mem. Amer. Math. Soc.*, vol. 59, no. 339, pp. i–vi and 99–130, 1986.

[4] D. Calegari, "Taut foliations leafwise branch cover S^2," *Algebraic Geometric Topology*, vol. 21, no. 5, pp. 2523 – 2541, 2021.

[5] D. Gabai, "Foliations and the topology of 3-manifolds," *Journal of Differential Geometry*, vol. 18, no. 3, pp. 445 – 503, 1983.