Dancing Planets and Modular Multiplication

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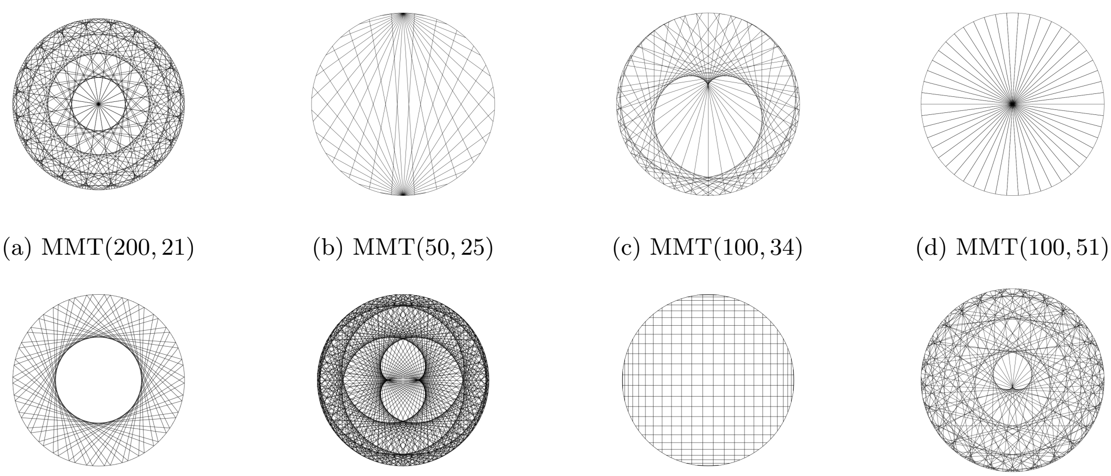
1 Modular Multiplication Tables



(a) MMT$(200, 21)$ (b) MMT$(50, 25)$ (c) MMT$(100, 34)$ (d) MMT$(100, 51)$

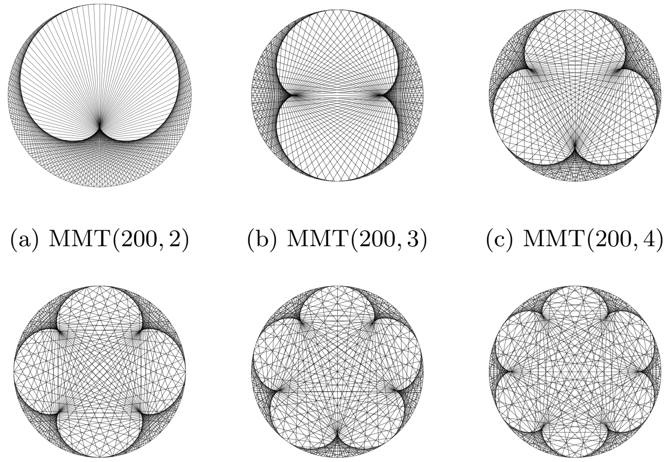
(e) MMT$(90, 31)$ (f) MMT$(400, 115)$ (g) MMT$(100, 49)$ (h) MMT$(206, 21)$

Figure 1: MMTs with various values for m and a.

Choose some natural number m; then, place m evenly spaced points around a circle, and label them 0 through $m-1$. Now choose an integer a and draw a directed chord from every point p to ap mod m. For example, let $m = 100$ and $a = 2$. What picture do you see?

Such an object is a *modular multiplication table*, denoted MMT(m, a). Figure 1 showcases the large variety of patterns achieved in this way. These objects appear throughout recreational mathematics and you may have seen them under another name— “string art”, “light caustics”, “spirographs”, and “curve stitching”, to name a few. One popular appearance is in a YouTube video by the channel *Mathologer*, “Times Tables, Mandelbrot, and the Heart of Mathematics” [1]. In fact, Mathologer’s video inspired the investigations which resulted in this paper.

A table MMT(m, a) is a finite set of chords on the circle. These chords maintain a direction even though we draw them without arrows. Although a can be any integer, note that MMT(m, a_1) and



(a) MMT$(200, 2)$ (b) MMT$(200, 3)$ (c) MMT$(200, 4)$

(d) MMT$(200, 5)$ (e) MMT$(200, 6)$ (f) MMT$(200, 7)$

Figure 2: MMTs with $m = 200$ and increasing a values

MMT(m, a_2) have the same set of chords if $a_1 \equiv a_2 \mod m$. Hence, we typically choose multipliers a with $0 \leq a < m$.

Big Question. *Given values for m and a, what pattern will* MMT(m, a) *create?*

Before reading on, I encourage the reader to explore these objects for themselves. There are many tools online to draw these pictures; one that I would recommend is made by Mathias Lengler and can be found at this link [2]. My own code to draw these objects can be found on my github page.

1.1 Some basic patterns

Let's start by fixing a small value for a and increasing m. A pattern emerges: for large enough m, the envelope of the chords is a curve with $a - 1$ petals. Figure 2 shows the first few curves in this sequence, called *epicycloids*. The curve for MMT(m, a) is made by rolling a circle of radius 1 around a fixed circle of radius $a - 1$ and tracking a point on the boundary of the outer circle. See Figure 3 for an example of this rolling circle construction with $a = 4$. The epicycloid with $a - 1$ petals is parameterized by the equations (1) with $\alpha = a$ and $\beta = 1$.

As we experiment with small values of a (approximately less than 20), we might be led to believe that the envelope of MMT(m, a) for sufficiently large m is always the epicycloid with $a - 1$ petals. We shall see that this would be a hasty conclusion. Playing around with higher multipliers unveils the variety of patterns showcased in Figure 1. We begin to suspect that the numerical relationship between m and a plays a key role in determining the pattern.

Question 1. *The three families of tables* MMT$(2a, a)$, MMT$(2a - 2, a)$, *and* MMT$(2a + 2, a)$ *each have a representative in Figure 1. Can you explain why these patterns appear?*

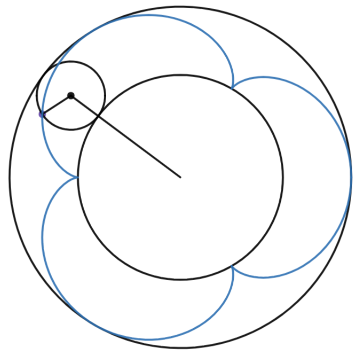


Figure 3: Constructing the epicycloid for $a = 4$. The smaller circle has radius $\frac{1}{5}$ and the larger circle has radius $\frac{3}{5}$.

Remark 1.1. Although it is often useful to think about modular multiplication tables as combinatorial objects, it will also be helpful for our purposes to define them geometrically. Parameterize the unit circle in $\mathbb{C}$ by $e^{2\pi it}$ for $t \in [0,1]$. Then $\mathrm{MMT}(m,a)$ is the set of chords with initial point $e^{2\pi it_k}$ and terminal point $e^{2\pi iat_k}$ with $t_k = \frac{k}{m}$ for all $k \in \{0,\ldots,m-1\}$.

1.2 Epicycloids and hypocycloids

Recall from Section 1.1 than an *epicycloid* is a curve constructed by rolling one circle around another fixed circle and tracking a point on the boundary of the outer circle. A *hypocycloid* is similar to an epicycloid but the rolling circle is on the inside of the fixed circle. We can realize both these curves with the following parametric equations.

$$\begin{aligned} x(t) &= \alpha\cos t + \beta\cos\left(\tfrac{\alpha}{\beta}t\right) \\ y(t) &= \alpha\sin t + \beta\sin\left(\tfrac{\alpha}{\beta}t\right) \end{aligned} \tag{1}$$

In the case of an epicycloid, both α and β should be positive real numbers. When $0 < \beta \leq \alpha$, the equations (1) describe a circle of radius β rolling around a circle of radius $\alpha - \beta$. If $0 < \alpha < \beta$, this physical interpretation does not make sense because we cannot have a circle with radius $\alpha - \beta$. However, switching the roles of α and β by setting $\alpha' = \beta$ and $\beta' = \alpha$ produces the same curve. This follows from a change of variables.

To draw a hypocycloid, we maintain $\alpha > 0$ but set $\beta < 0$. Then the equations (1) describe the hypocycloid with fixed circle radius $\alpha - \beta$ and rolling circle radius $|\beta|$.

In this way, we can realize epicycloids and hypocycloids using the same set of parametric equations. For our purposes, we will only be concerned with cases where α and β are integers, because we are interested in periodic curves.

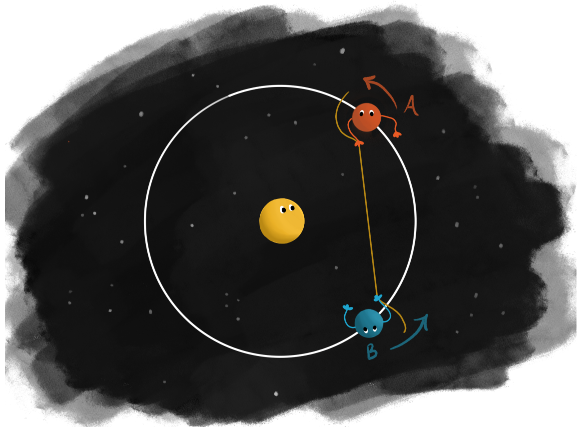


Figure 4: Two planets on the same circular orbit with a stretchy tether between.

2 Dancing Planets

Imagine two planets on the same circular orbit where planet B is moving twice as fast as planet A. Held between them is an infinitely stretchy tether always pulled taught. As the planets orbit, we watch from above and take a picture 100 times at regular intervals during planet A's orbit. The result? A picture which looks very similar to $\mathrm{MMT}(100, 2)$. (See Figure 4)

Definition 2.1. We denote a *planet dance* by $\mathcal{P}(\alpha, \beta)$ where $\alpha, \beta \in \mathbb{Z}$ are integers. These integers define a set of directed chords of the circle with initial point at $e^{2\pi i\alpha t}$ and terminal point at $e^{2\pi i\beta t}$ for all $t \in [0, 1]$.

When the planet dance $\mathcal{P}(\alpha, \beta)$ is $\mathcal{P}(0, 0)$, $\mathcal{P}(0, 1)$, $\mathcal{P}(1, 0)$, or $\gcd(\alpha, \beta) = 1$, we say that $\mathcal{P}(\alpha, \beta)$ is in *reduced form*.

Definition 2.2. If α and β have the same sign, then $\mathcal{P}(\alpha, \beta)$ is a *positive planet dance*. If α and β have opposite signs, then $\mathcal{P}(\alpha, \beta)$ is a *negative planet dance*. We will typically assume that $\alpha \geq 0$, so that the sign of the planet dance is determined by the sign of β.

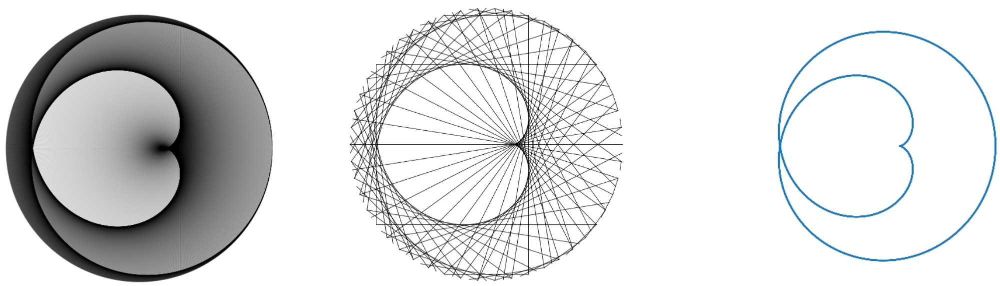
For a positive planet dance $\mathcal{P}(\alpha, \beta)$, the chords envelope a recognizable epicycloid. Figure 5 shows both $\mathcal{P}(3, 2)$ and the corresponding epicycloid: this curve is described by the equations (1) with $\alpha = 3$ and $\beta = 2$.

Upon first inspection, negative planet dances do not share this correspondence. However, the pattern is still there, though a bit hidden. To see the correspondence with an hypocycloid, we need to extend the chords beyond the circle. The curve will be given by a circle with radius $|\beta|$ rolling inside a circle with radius $\alpha - \beta$.[1] Figure 6 shows $\mathcal{P}(5, -3)$ and its corresponding hypocycloid.

In both the positive and the negative case, the curve (epicycloid or hypocycloid) will be given by the parametric equations (1) in Section 1.2 (see Theorems 25 and 27 in [3] for a proof).

Definition 2.3 (m-sampling of a planet dance)**.** Let $\mathcal{P}(\alpha, \beta)$ be a planet dance and m be a positive integer. An *m-sampling* of $\mathcal{P}(\alpha, \beta)$ is the finite set of chords for $t = 0, \frac{1}{m}, \frac{2}{m}, \ldots, \frac{m-1}{m}$. We denote this set $\mathcal{S}(\alpha, \beta, m)$.

[1] We are assuming that $\alpha \geq 0$ as discussed in Definition 2.2



(a) $\mathcal{P}(3,2)$

(b) 100-sample of $\mathcal{P}(3,2)$

(c) Epicycloid formed by radius 2 circle rolling around radius 1 circle.

Figure 5: A planet dance and its 100-sample

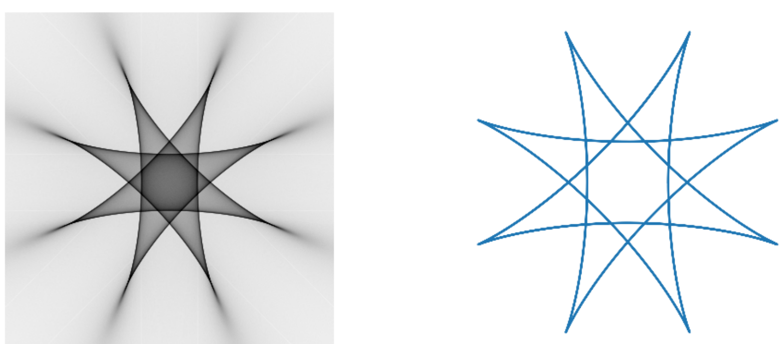


Figure 6: $\mathcal{P}(5,-3)$ and the corresponding hypocycloid

A sampling of a planet dance looks rather like a modular multiplication table (see Figure 5 for an example). But does this visual correlation point to an actual mathematical correspondence?

Question 2. *For every* $\mathrm{MMT}(m, a)$ *are there* α *and* β *such that* $\mathrm{MMT}(m, a) = \mathcal{S}(\alpha, \beta, m)$*? And, for each* $\mathcal{S}(\alpha, \beta, m)$*, can we find* a *such that* $\mathcal{S}(\alpha, \beta, m) = \mathrm{MMT}(m, a)$*?*

The first of these questions has an immediate affirmative answer. The modular multiplication table $\mathrm{MMT}(m, a)$ is exactly the same set of chords as $\mathcal{S}(1, a, m)$. When the initial end of the chord has traveled distance p, the terminal end will have traveled a distance of ap. Thus, each chord in $\mathcal{S}(1, a, m)$ can be constructed by multiplying the initial point p by a. We will call planet dances of the form $\mathcal{P}(1, \beta)$ *integral dances.*

Lemma 2.4 (Fundamental Correspondence)**.** *The modular multiplication table* $\mathrm{MMT}(m, a)$ *is an* m*-sampling of the integral planet dance* $\mathcal{P}(1, a)$*.*

The answer to our second question is not so simple. If we have a planet dance $\mathcal{P}(\alpha, \beta)$ with $\alpha > 1$ and $\gcd(\alpha, \beta) = 1$, the multiplier of the corresponding modular multiplication table should be $\frac{\beta}{\alpha}$. But in our definition of modular multiplication tables, we do not allow non-integral multipliers. Of course, we could change the definition, but we will see that there is a way forward without this restructuring.

Remark 2.5. Other sources have taken the route of defining modular multiplication tables using non-integer multipliers. See [2] for an example.

Why do we hope that we can use the current definition of modular multiplication tables to represent samplings of non-integral planet dances? Empirical evidence. Figure 1c depicts $\mathrm{MMT}(100, 34)$ and we notice that it looks very similar to $\mathcal{P}(3, 2)$ in Figure 5a. In fact, any MMT of the form $\mathrm{MMT}(3a - 2, a)$ will envelop this same epicycloid curve[2]. What is the deeper reason that these two patterns look the same?

3 Introducing Topology

3.1 Planet dances as paths on a torus

A directed chord on the circle is uniquely determined by the position of the two endpoints—planet A and planet B. Hence, the space of all such possible chords is $\mathbb{S}^1 \times \mathbb{S}^1 = \mathbb{T}^2$, a torus!

A point on the torus gives a single chord on the circle, and a path on the torus gives a continuous family of chords. For a planet dance $\mathcal{P}(\alpha, \beta)$, we know planet A and planet B are moving at constant speeds α and β respectively. So all chords in $\mathcal{P}(\alpha, \beta)$ form a linear closed path on the flat torus. These loops are commonly used to represent torus knots. Figure 7 shows two examples of these linear paths on the torus.

We will parameterize the circle with $e^{2\pi i t}$ for $t \in [0, 1]$. We can represent $\mathbb{T}^2$ as the unit square in $\mathbb{R}^2$ with opposite sides identified or, equivalently, $\mathbb{T}^2 \cong \mathbb{R}^2/\mathbb{Z}^2$. Thus, the line in $\mathbb{R}^2/\mathbb{Z}^2$ which corresponds to $\mathcal{P}(\alpha, \beta)$ is given by

$$x = \alpha t, \quad y = \beta t, \quad \text{ or, if } \alpha \neq 0, \quad y = \tfrac{\beta}{\alpha} x.$$

We will identify the planet dance $\mathcal{P}(\alpha, \beta)$ with this line on the torus, and depending on the context, refer to this line as $\mathcal{P}(\alpha, \beta)$.

[2] This example is one member in an infinite family that initiated this research project. I discuss this story and the family of tables in Section 4.

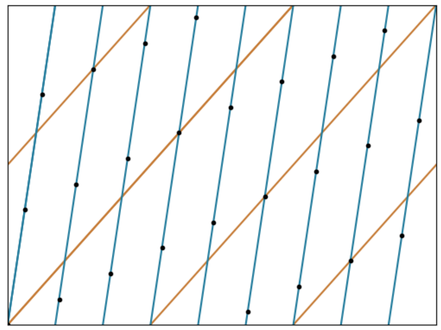


Figure 7: An example of linear loops on the torus. The blue lines depict $y = \frac{3}{2}x$ which corresponds with $\mathcal{P}(2,3)$ and the orange lines depicts $y = 9x$ which corresponds to $\mathcal{P}(1,9)$. The black dots indicate that $\mathcal{P}(1,9)$ is sampled at a rate of 25, corresponding with $\mathrm{MMT}(25,9)$.

Remark 3.1 (On direction). Swapping the values of α and β in a planet dance will not make a difference in the picture, but it does change the direction of each chord and change the slope of the corresponding line in $\mathbb{R}^2/\mathbb{Z}^2$. Moving forward, we will maintain the direction of chords as this inherently matters for modular multiplication tables. In an MMT, we think of the initial point of the chord being multiplied by a to get the terminal point of the chord. That is, we would like $\mathrm{MMT}(m,a)$ to correspond with $\mathcal{S}(1,a,m)$ and not $\mathcal{S}(a,1,m)$.

If a planet dance is a linear loop on the torus, then a sampling of the planet dance is a set of equally spaced points along the loop. As the sampling becomes more frequent, the set of points more closely approximates the loop and the discrete sampling picture becomes a better representation of the continuous planet dance.

3.2 Aliasing in planet dances

Aliasing is a pervasive topic in signal processing. The idea is best demonstrated with sine waves. Suppose that a computer is storing discrete-time samples of a musical note, represented as a sine wave. If the sampling rate is too low, then when the computer plays back the note from its discrete samples, the listener hears it as a lower note, i.e. with a lower frequency. This happens because there is more "natural" wave that fits the samples. The two waves—the original sine wave and the lower frequency alias—intersect exactly at the sample points. An example is shown in Figure 8. Common examples of aliasing are Moiré patterns and the "wagon wheel" effect in cinema. In signal processing, aliasing is often seen as an unfortunate reality, and research into "anti-aliasing filters" is common. However, this phenomenon is why we see such a large variety of patterns among modular multiplication tables. This will be the focus for the rest of the paper.

Just as one sine wave can "alias" another when we take a discrete sampling, one planet dance can alias another. The key to this aliasing is the points of intersection. Suppose that the lines corresponding to planet dances $\mathcal{P}_1$ and $\mathcal{P}_2$ intersect m times in $\mathbb{R}^2/\mathbb{Z}^2$. These intersection points occur at regular intervals because both lines have constant rational slopes. Then the m-sampling of $\mathcal{P}_1$ and the m-sampling of $\mathcal{P}_2$ are the same set of chords and will look identical. Since we know

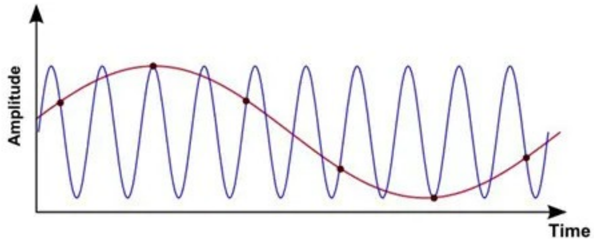


Figure 8: When a higher frequency sine wave is under-sampled, it can appear as a lower frequency sine wave where the sampling points are exactly the points of intersection. Image from [4].

how to achieve m-samplings of integral planet dances with MMTs, this gives us a way to sample any planet dance with an MMT by finding an integral "alias".

Lemma 3.2 (Planet dance intersection)**.** *Let lines ℓ_1 and ℓ_2 in $\mathbb{R}^2/\mathbb{Z}^2$ be given by*

$$\ell_1(t) = (\alpha t, \beta t) \quad \textit{and} \quad \ell_2(t) = (\gamma t, \delta t)$$

such that $\gcd(\alpha, \beta) = \gcd(\gamma, \delta) = 1$. *Then ℓ_1 and ℓ_2 will intersect $|\alpha\delta - \beta\gamma|$ times and will do so at regular intervals.*

This is a standard result often applied to intersection numbers of torus knots in $\mathbb{T}^2$. By restating Lemma 3.2, we can relate non-integral planet dances to modular multiplication tables.

Corollary 3.3. *Suppose that $\mathcal{P}(\alpha, \beta)$ and $\mathcal{P}(\gamma, \delta)$ are two planet dances in reduced form and let $m = |\alpha\delta - \beta\gamma|$. Then $\mathcal{S}(\alpha, \beta, m) = \mathcal{S}(\gamma, \delta, m)$.*

Proposition 3.4. *Given a planet dance $\mathcal{P}(\alpha, \beta)$ in reduced form and $a \in \mathbb{Z}$, let $m = |\alpha a - \beta|$. Then the modular multiplication table* $\mathrm{MMT}(m, a)$ *will be an m-sampling of $\mathcal{P}(\alpha, \beta)$.*

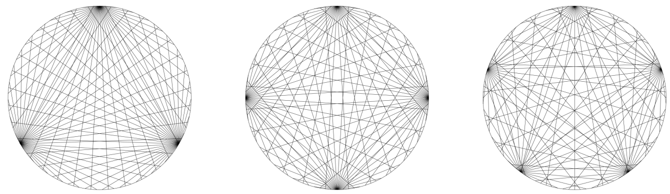
To address an inverse of Proposition 3.4, a sampling of a planet dance $\mathcal{S}(\alpha, \beta, m)$ can be realized as a modular multiplication table exactly when $m = |\alpha a - \beta|$ for some positive integer a. This has a solution exactly when $m \equiv \pm\beta \mod \alpha$.

If you choose a general planet dance $\mathcal{P}(\alpha, \beta)$, and a desired approximate sampling rate, then I can give you a modular multiplication table $\mathrm{MMT}(m, a)$ with m "close" to your sampling rate so that $\mathrm{MMT}(m, a)$ is an m-sampling of $\mathcal{P}(\alpha, \beta)$. How "close" depends on α—we choose the nearest m such that $m \equiv \pm\beta \mod \alpha$.

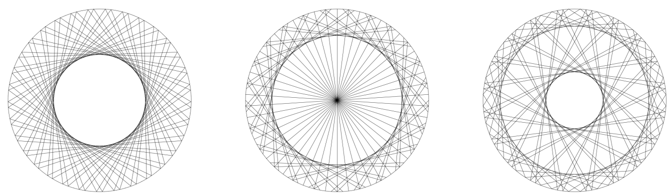
So we have answered Question 2 for all planet dances. Given any planet dance $\mathcal{P}(\alpha, \beta)$, we can find a corresponding family of MMTs which are all samplings of $\mathcal{P}(\alpha, \beta)$. This also reveals why certain MMTs exhibit patterns that are very different from the integral planet dance given by the Fundamental Correspondence Lemma. When $\mathrm{MMT}(m, a)$ is viewed as an m-sampling of $\mathcal{P}(1, a)$, if m is an "under-sampling" then $\mathrm{MMT}(m, a)$ will alias as a sampling of another planet dance. But this correspondence is only the tip of the iceberg.

4 Overlaying Patterns

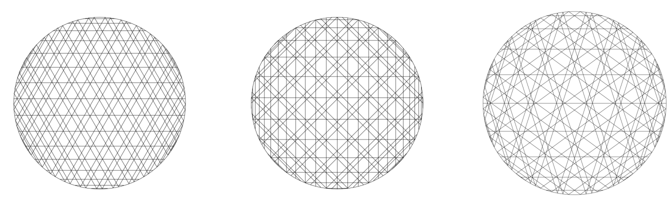
We return to the patterns in modular multiplication tables that we noticed in Section 1. Assuming that m is even, we observed that tables of the form $\mathrm{MMT}(m, \frac{m}{2})$ or $\mathrm{MMT}(m, \frac{m}{2} + 1)$ or



(a) MMT$(99, 33)$ (b) MMT$(100, 25)$ (c) MMT$(100, 20)$



(d) MMT$(99, 34)$ (e) MMT$(100, 26)$ (f) MMT$(100, 21)$



(g) MMT$(99, 32)$ (h) MMT$(100, 24)$ (i) MMT$(100, 19)$

Figure 9: The top row shows examples of tables of the form $(m, \frac{m}{b})$ with $b = 3, 4$, and 5 from left to right. The second row shows examples of tables of the form $(m, \frac{m}{b} + 1)$ with the same b values. The last row shows tables of the form $(m, \frac{m}{b} - 1)$ again with the same values for b.

MMT$(m, \frac{m}{2} - 1)$ each display a distinct design which holds for all such m (recall Question 1). A natural generalization is to replace '2' with any natural number b. That is, supposing that m is divisible by b, we consider three classes of tables:[3]

$$\mathrm{MMT}\left(m, \tfrac{m}{b}\right), \qquad \mathrm{MMT}\left(m, \tfrac{m}{b} + 1\right), \quad \text{and} \quad \mathrm{MMT}\left(m, \tfrac{m}{b} - 1\right).$$

The designs for each of these classes are generalizations of those when $b = 2$. Examples of these of tables are shown in Figure 9.

As I was considering these families of tables, I wondered if we could define a similar kind of MMT without the assumption that m is divisible by b. I decided to look at MMTs where $b \nmid m$ and $a = \lceil \frac{m}{b} \rceil$ or $a = \lfloor \frac{m}{b} \rfloor$, since I still wanted a to be an integer. I was amazed to find many clear patterns in tables of this form. Figure 10 shows tables of the form MMT$(m, \lceil \frac{m}{b} \rceil)$. The rows are indexed by the value for b and the columns are indexed by the remainder of m mod b. I encourage the reader to take a minute and find patterns within this "table of tables".

[3]One might want to describe these families as MMT(ba, a) and MMT$(ba \pm b, a)$ to avoid the assumption that $b|m$. However, in order to keep the narrative of discovery coherent, we will keep the above perspective for now.

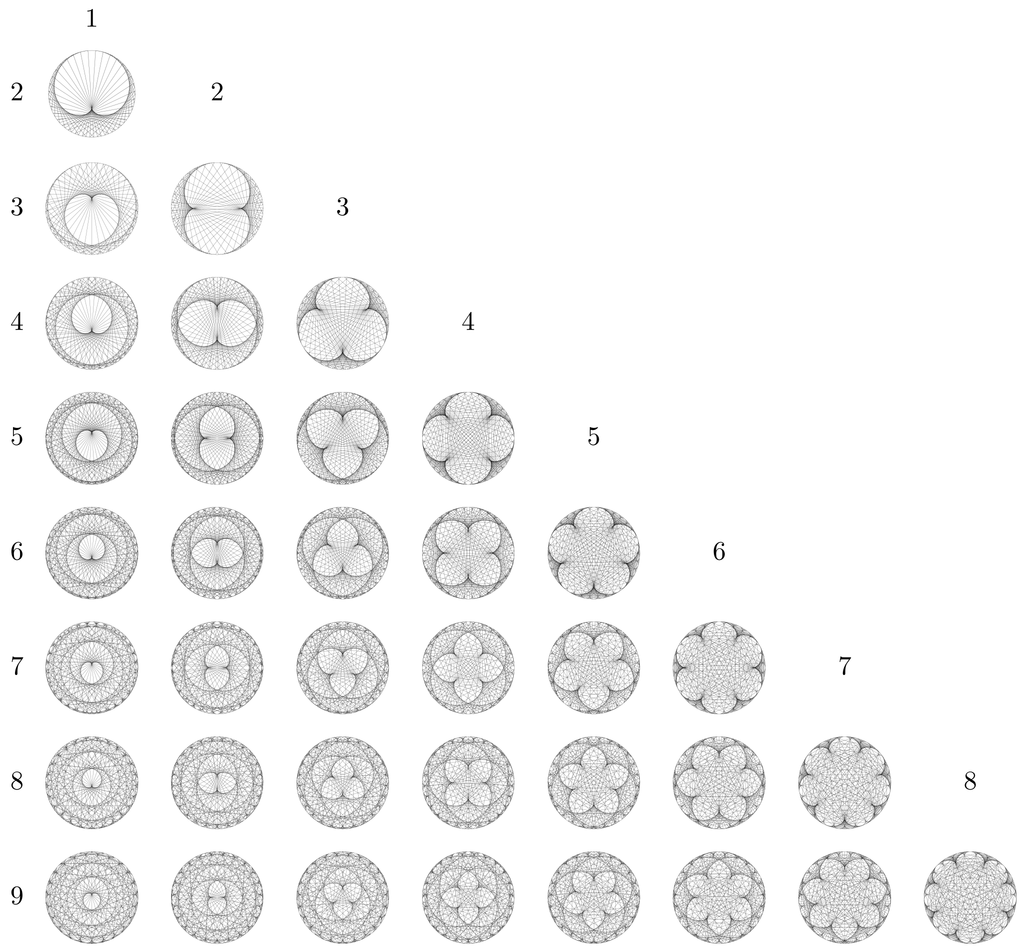
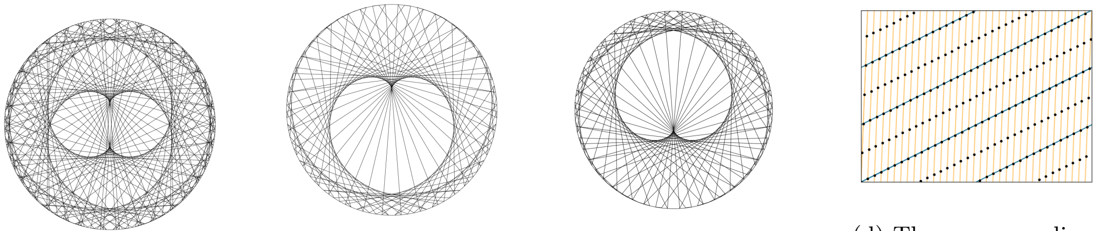


Figure 10: A "table of tables"! Each table is of the form $(m, \lceil \frac{m}{b} \rceil)$ for some positive integers with $b < m$. The rows are indexed by the value for b and the columns are indexed by r where $m \equiv r \mod b$. We have taken m to be relatively large so as to see the pattern clearly. Most values for m are approximately 100.



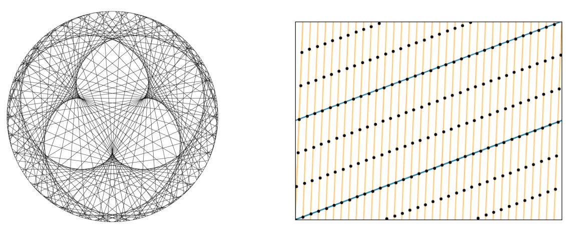
(a) MMT$(206, 35)$

(b) All chords with initial point even.

(c) All chords with initial point odd.

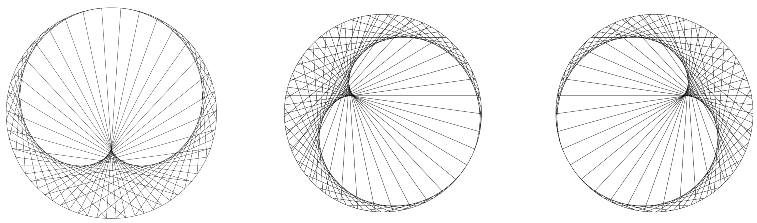
(d) The corresponding planet dance and its discrete sampling.

Figure 11: MMT$(206, 35)$ “reduces” to two copies of MMT$(103, 35)$, one rotated halfway around the circle. Figure 11d depicts the corresponding linear torus loops. The orange lines represent $\mathcal{P}(1, 35)$, with the black dots showing a sampling at rate 206. The blue lines represent $\mathcal{P}(3, 2)$.



(a) MMT$(207, 35)$

(b) The corresponding planet dance and its discrete sampling.



(c) All chords with initial point $p \equiv 0 \mod 3$

(d) All chords with initial point $p \equiv 1 \mod 3$

(e) All chords with initial point $p \equiv 2 \mod 3$.

Figure 12: MMT$(207, 35)$ “reduces” to three copies of MMT$(69, 35)$. Figure 12b depicts the corresponding linear torus loops. The orange lines represent $\mathcal{P}(1, 35)$, with the black dots showing a sampling at rate 207. The blue lines represent $\mathcal{P}(2, 1)$.

I discovered these patterns before considering MMTs from the perspective of planet dances. But now, with this other perspective, the mysteries are unraveled. Let $m \equiv r \mod b$. Then we can write $a = \lceil \frac{m}{b} \rceil$ as

$$a = \frac{m + (b - r)}{b}.$$

If we do some algebraic rearranging, then we have the following relationship.

$$\begin{aligned} ba &= m + b - r \\ ba - (b - r) &= m \end{aligned}$$

With our knowledge about the intersection number of linear loops on the torus, we recognize that the above equation says that $\mathcal{P}(b, b-r)$ and $\mathcal{P}(1, a)$ will intersect m times. However, there is one missing link: this is only true if both planet dances are in reduced form, i.e. if $\gcd(b, b - r) = 1$. In this case, $\mathrm{MMT}(m, a)$ will be an m-sampling of $\mathcal{P}(b, b-r)$ by Proposition 3.4. We have already seen one example of this case: when $b = 3$ and $r = 1$. If we set $a = 34$, then $m = 3 \cdot 34 - 2 = 100$. Following the discussion in Section 3.2, $\mathrm{MMT}(100, 34)$ is a 100-sampling of $\mathcal{P}(3, 2)$.

Corollary 4.1. *Let m and b be positive integers with $b < m$ such that $m \equiv r \mod b$ for $0 < r < b$. If $\gcd(r, b) = 1$, then $\mathrm{MMT}(m, \lceil \frac{m}{b} \rceil)$ will be an m-sampling of $\mathcal{P}(b, b - r)$.*

However, it may be the case that $\gcd(b, b - r) = d > 1$. Upon examination of the "table of tables", one might notice that many of the images look like overlays of other tables. For example, when $r = 2$ and $b = 6$, the design looks like two copies of the pattern when $r = 1$ and $b = 3$, but one copy is rotated halfway around the circle. When I first found this family of tables, I noticed this visually and I decided to test my idea experimentally. Figure 11 shows a member of the $r = 2$, $b = 6$ family ($\mathrm{MMT}(206, 35)$) along with two subsets of this MMT. The first shows only the chords originating from even points (Figure 11b) and the second shows only the chords originating from odd points (Figure 11c). Another example of this overlay phenomenon is shown in Figure 12.

Every example that I tested gave the same result. When $\gcd(b, b-r) = d$, the table $\mathrm{MMT}(m, \lceil \frac{m}{b} \rceil)$ consisted of d copies of $\mathrm{MMT}(\frac{m}{d}, \lceil \frac{m}{b} \rceil)$, rotated symmetrically around the circle. I managed to devise a number theoretic proof of this fact but it was not very illuminating. With our topological perspective, we can see what is going on more clearly. When we draw the corresponding linear torus loops for the examples in Figures 11 and 12, we see that the sampling points trace out copies of $\mathcal{P}(b, b - r)$ which are shifted up. Although we do not get these "phantom copies" of linear loops from the planet dance construction, they do appear by sampling.

The following theorem addresses a general case of overlaying tables, which we will then apply to our specific example.

Theorem 4.2 (Overlaying planet dances)**.** *Let $\mathcal{P}(\alpha, \beta)$ be a planet dance in reduced form with $\alpha \neq 0$ and let $a \in \mathbb{Z}$ be any integer. For $m = |\alpha a - \beta|$ we have $\mathcal{S}(\alpha, \beta, m) = \mathcal{S}(1, a, m)$ by Proposition 3.4. Then, for any positive integer d, $\mathcal{S}(1, a, dm)$ consists of d sets $\mathcal{S}(\alpha, \beta, m)$, each one rotated by $\frac{k}{(\alpha - \beta)d}$ around the circle for $k = 0, 1, \ldots, d - 1$.*

Proof. We will think of all points and lines as lying in $\mathbb{R}^2/\mathbb{Z}^2$ but we will use coordinates for $\mathbb{R}^2$ to simplify notation. Recall that $\mathcal{P}(\alpha, \beta)$ is given by a line in $\mathbb{R}^2/\mathbb{Z}^2$ with the parametric equations

$$x = \alpha t, \quad y = \beta t, \quad \text{or} \quad y = \tfrac{\beta}{\alpha} x \quad \text{since } \alpha \neq 0.$$

Let $Q = \{q_0, q_1, \ldots, q_{dm-1}\} \subset T^2$ be the set of sample points $\mathcal{S}(1, a, dm)$, ordered so that $q_j = (\frac{j}{dm}, a\frac{j}{dm})$. For each $0 \leq k \leq d-1$, we define $Q_k := \{q_j \in Q \mid j \equiv k \mod d\}$. Notice that $Q_0 = \{q_0, q_d, q_{2d}, \ldots q_{(m-1)d}\}$ is the set of sample points $\mathcal{S}(1, a, m)$ and so they all lie on the line $y = \frac{\beta}{\alpha}x$ since $\mathcal{S}(1, a, m) = \mathcal{S}(\alpha, \beta, m)$. We claim that all points in Q_k lie on the line given by

$$x = \alpha t, \quad y = \beta t + \frac{k}{d\alpha}. \tag{2}$$

Note that the points in Q_k are of the form q_{id+k} for some $0 \leq i \leq m-1$. Since the sampling is regular, we have

$$q_{id+k} = q_{id} + \tfrac{k}{d}(q_d - q_0),$$

where the points are treated as vectors with addition and scalar multiplication. In particular, the vector that we add to q_{id} is invariant of i. Since Q_0 is a set of colinear points lying on the line $y = \frac{\beta}{\alpha}x$, and translating a line leaves the slope invariant, we conclude that Q_k lies on a line of the form

$$y = \frac{\beta}{\alpha}x + C_k$$

where C_k is a constant. We then use the Triangle Proportionality theorem to find that the y-intercept of the line is $\frac{k}{d\alpha}$. See Figure 13.

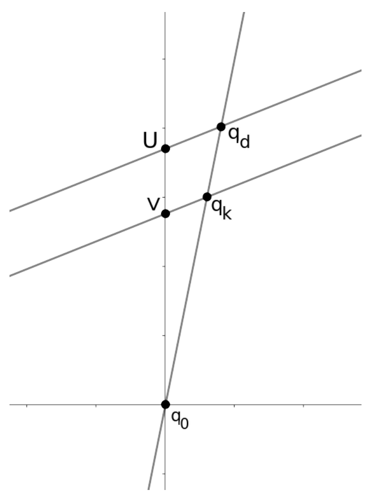


Figure 13: The point q_k lies between q_0 and q_d on the line $y = ax$. The Triangle Proportionality Theorem says that $\frac{\|U-V\|}{\|V\|} = \frac{\|q_d - q_k\|}{\|q_k\|}$. Since $U = (0, \frac{1}{\alpha})$, we find that $V = \frac{k}{d\alpha}$.

The last step is to show why the line given by (2) is a copy of $\mathcal{P}(\alpha, \beta)$ rotated by $\frac{k}{d(\alpha-\beta)}$. We notice that every planet dance $\mathcal{P}(\alpha, \beta)$ starts at $t = 0$ with both planets in the same position. This is exactly an intersection with the line $\mathcal{P}(1, 1)$. But since (2) has a non-zero y-intercept, the corresponding planets start at different positions at $t = 0$. So the rotation amount of the corresponding planet dance is determined by the point of intersection between the line (2) with

$\mathcal{P}(1,1)$. We solve for such a t.

$$\begin{aligned}\alpha t &= \beta t + \frac{k}{d\alpha}\\ (\alpha-\beta)t &= \frac{k}{d\alpha}\\ t &= \frac{k}{d\alpha(\alpha-\beta)}\end{aligned}$$

And then we find $x = y = \frac{k}{d(\alpha-\beta)}$ at this value for t. $\square$

Now we apply this general theory of overlays to extend Corollary 4.1 to cases where $\gcd(r,b) > 1$. We give an explicit description for tables of the form $\mathrm{MMT}(m, \lceil \frac{m}{b} \rceil)$ and $\mathrm{MMT}(m, \lfloor \frac{m}{b} \rfloor)$.

Proposition 4.3. *Let m and b be positive integers with $b < m$ such that $b \nmid m$. Let $m \equiv r \mod b$ for $0 < r < b$ and let $d = \gcd(r,b)$. Then:*

- *the table $\mathrm{MMT}(m, \lceil \frac{m}{b} \rceil)$ is composed of d sets $\mathcal{S}(\frac{b}{d}, \frac{b-r}{d}, \frac{m}{d})$, each one rotated by $\frac{k}{r}$ around the circle for $0 \le k \le d-1$;*
- *the table $\mathrm{MMT}(m, \lfloor \frac{m}{b} \rfloor)$ is composed of d sets $\mathcal{S}(\frac{b}{d}, \frac{-r}{d}, \frac{m}{d})$, each one rotated by $\frac{k}{r}$ around the circle for $0 \le k \le d-1$.*

5 Wrapping up and moving forward

Let's now return to the big question at the beginning of this article:

Given m and a, can we know what $\mathrm{MMT}(m,a)$ will look like?

The number theoretic answer is to find integers α and β such that $m = |\alpha a - \beta|$. However, there are many solutions and most will not be the most natural answer. Instead, we can graph the points $\mathcal{S}(1,a,m)$ in $\mathbb{T}^2$ and determine the most natural way to connect the dots. We then find the "best choice" for α and β from the slope of this line and we can predict that $\mathrm{MMT}(m,a)$ will look like the corresponding epicycloid or hypocycloid. Although, it seems this is not any easier than simply asking a computer to draw $\mathrm{MMT}(m,a)$!

To close out, we return to the variety of patterns shown in Figure 1. I will leave you with pictures of these tables along with the corresponding sampled torus loop. Does this new perspective shed light on the patterns in $\mathrm{MMT}(m,a)$?

Remark 5.1. If you want to play around with more of these pictures, you can download my Python script here. It will let you draw MMTs, planet dances, linear torus loops, and epicycloids. Enjoy!

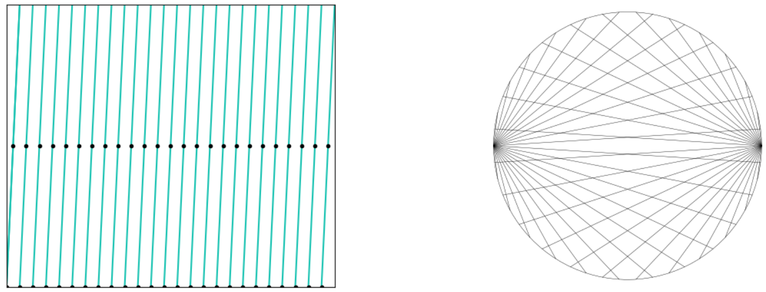


Figure 14: A table of the form $\mathrm{MMT}(2a, a)$ along with $\mathcal{P}(1, a)$ sampled at rate $2a$.

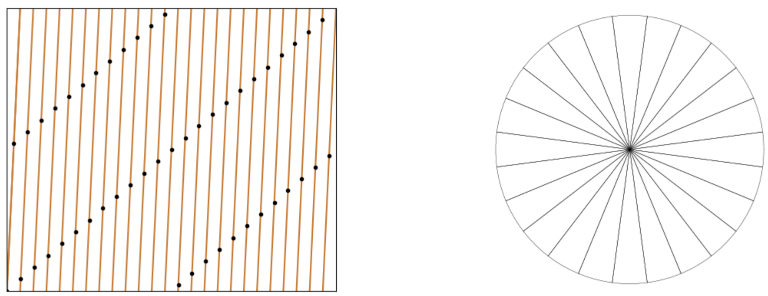


Figure 15: A table of the form $\mathrm{MMT}(2a-2, a)$ along with $\mathcal{P}(1, a)$ sampled at rate $2a-2$.

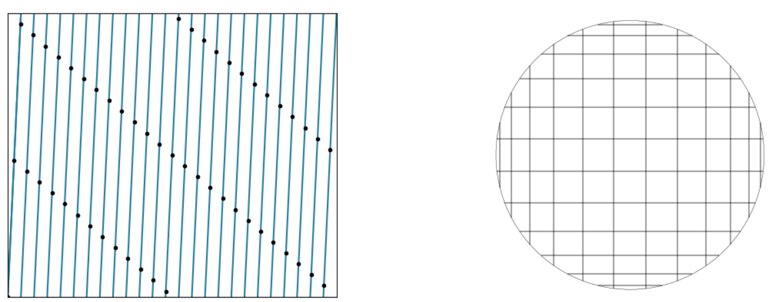


Figure 16: A table of the form $\mathrm{MMT}(2a+2, a)$ along with $\mathcal{P}(1, a)$ sampled at rate $2a+2$.

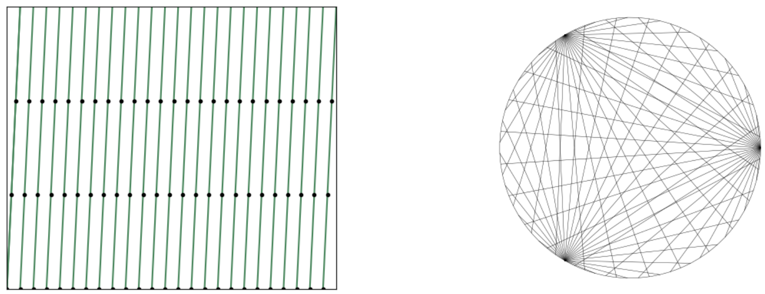


Figure 17: A table of the form $\mathrm{MMT}(3a, a)$ along with $\mathcal{P}(1, a)$ sampled at rate $3a$.

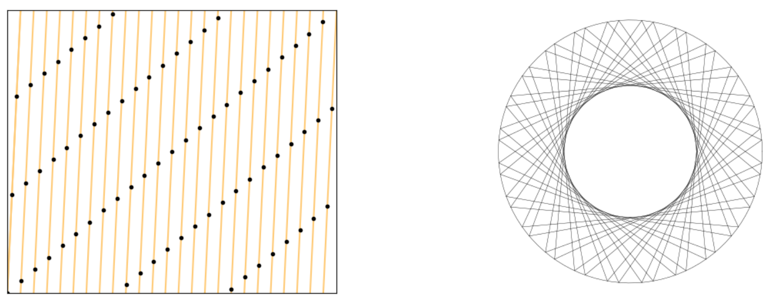


Figure 18: A table of the form $\mathrm{MMT}(3a - 3, a)$ along with $\mathcal{P}(1, a)$ sampled at rate $3a - 3$.

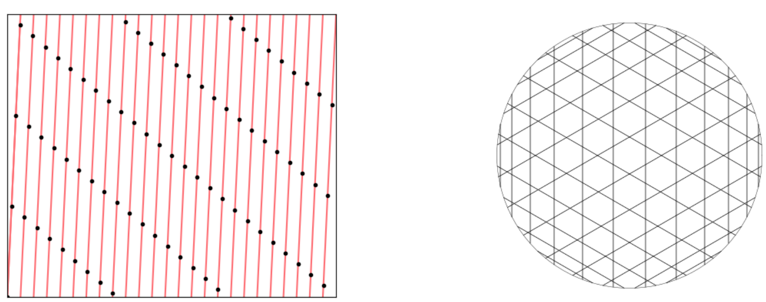


Figure 19: A table of the form $\mathrm{MMT}(3a + 3, a)$ along with $\mathcal{P}(1, a)$ sampled at rate $3a + 3$.

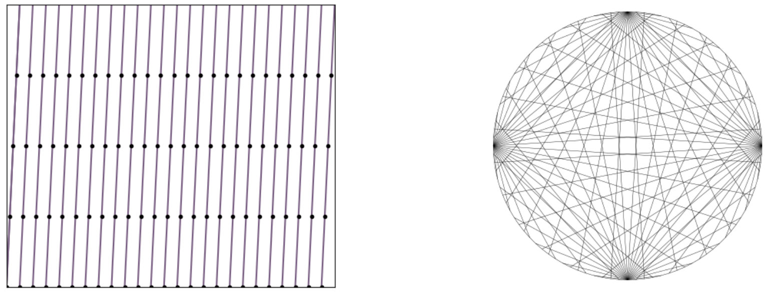


Figure 20: A table of the form $\mathrm{MMT}(4a, a)$ along with $\mathcal{P}(1, a)$ sampled at rate $4a$.

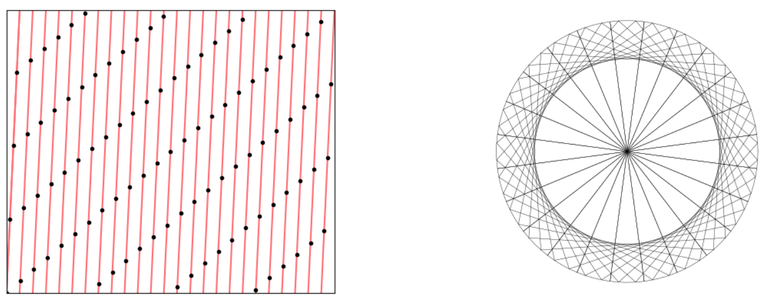


Figure 21: A table of the form $\mathrm{MMT}(4a-4, a)$ along with $\mathcal{P}(1, a)$ sampled at rate $4a-4$.

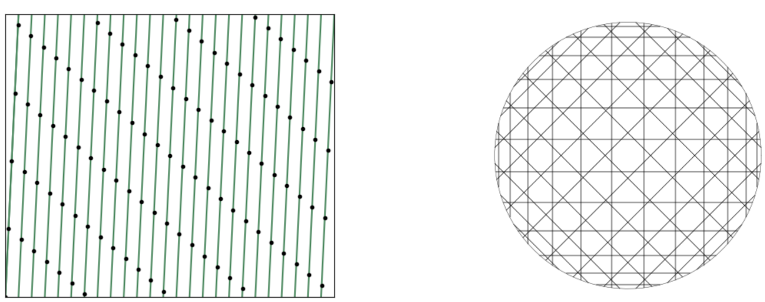


Figure 22: A table of the form $\mathrm{MMT}(4a+4, a)$ along with $\mathcal{P}(1, a)$ sampled at rate $4a+4$.

References

[1] B. Polster, "Times tables, mandelbrot, and the heart of mathematics." Mathologer, YouTube, Nov 2015.

[2] M. Lengler, "Timestablewebgl." https://lengler.dev/TimesTableWebGL/.

[3] M. Bouthillier, "Representations of epitrochoids and hypotrochoids," Master's thesis, Dalhousie University, April 2018.

[4] "Dynamic signal analysis review: Part 3 – aliasing frequency." https://dataphysics.com/blog/dynamic-signal-analysis/dynamic-signal-analysis-review-part-3-aliasing/, Jan 2016. Data Physics Corporation.

[5] A. I. Robold and P. Yff, "From curve stitching to epicycloids," *School Science and Mathematics*, vol. 96, no. 7, pp. 350–354, 1996.

[6] J. A. Boyle, "Using rolling circles to generate caustic envelopes resulting from reflected light," *The American Mathematical Monthly*, vol. 122, no. 5, pp. pp. 452–466, 2015.

[7] M. Hopkins, "The envelope of a line moving between two concentric circles," *The Mathematics Teacher*, vol. 43, no. 6, pp. 264–268, 1950.

[8] A. J. Simoson, "An envelope for a spirograph," *College Mathematics Journal*, vol. 28, pp. 134–139, 1997.

[9] B. Haran and M. Henderson, "The strange orbit of earth's second moon (plus the planets)." Numberphile, YouTube, Sep 2021.