

AN EXPLICIT LEAFWISE BRANCHED COVERING FOR A TRANSVERSE PSEUDO-ANOSOV FLOW

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ABSTRACT. We construct a specific branched submersion for a 3-manifold with a taut codimension one foliation transverse to a pseudo-Anosov flow. This map is a leafwise branched covering for the foliation: the restriction to each leaf is a branched covering. Moreover, its critical locus is a set of closed orbits of the flow, containing all the singular orbits. The construction depends only on the pseudo-Anosov flow and is done via an associated veering triangulation. We also develop framework for branched submersions, homotopies through branched immersions, and taut cone fields. In the last section, we work out an explicit example for the figure eight knot complement.

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1. INTRODUCTION

Let M be a 3-manifold with a codimension one foliation $\mathcal{F}$. In [6], Calegari proves that $\mathcal{F}$ is taut if and only if there is a continuous map $\psi : M \rightarrow S^2$ such that the restriction of ψ to each leaf $\lambda \in \mathcal{F}$ is a branched covering. We call such a map a *leafwise branched covering*.

Many leafwise branched coverings exist for a given foliation. However, it is a difficult problem to determine the “simplest” such map. To approach this question, we are interested in gathering more explicit examples of leafwise branched coverings, especially relating to other low-dimensional structures on the 3-manifold.

In this paper we address the situation where a taut foliation $\mathcal{F}$ is transverse to a pseudo-Anosov flow φ . We construct an explicit leafwise branched covering for $\mathcal{F}$ such that the critical locus is a union of closed orbits of φ containing all the singular orbits.

Theorem A. *Let M be a compact 3-manifold, possibly with torus boundary, with a taut foliation $\mathcal{F}$ and a transitive pseudo-Anosov flow φ transverse to $\mathcal{F}$. Then there is a leafwise branched covering $\psi : M \rightarrow S^2$ with its critical locus a union of closed orbits of φ containing all the singular orbits.*

Independent of the foliation. The construction of ψ depends only on the pseudo-Anosov flow; if two foliations are transverse to the same pseudo-Anosov flow, then the same map $\psi : M \rightarrow S^2$ will be a leafwise branched covering for both. We then refer to ψ as a *branched submersion* to emphasize its independence from any specific foliation. Such a map $\psi : M \rightarrow S$ from a 3-manifold to a surface is a submersion outside of a *critical locus/singular set* $X(\psi)$ around which it is given by a continuous family of branched immersions. If a surface $\Sigma \hookrightarrow M$ is transverse to the fibers of ψ and to $X(\psi)$, then up to a small isotopy, $\psi|_{\Sigma}$ is a branched immersion. (This isotopy may be needed to ensure that Σ satisfies a rather technical condition: for all $q \in X(\psi)$, $T_q \Sigma \subset \ker D_q \psi$. See Section 3.5.) From this perspective, we make a connection to cone fields defined using a Lorentzian metric. In Section 5, we define *taut cone fields* and we use them to express this transversality condition.

Proof outline. The key tool for the construction in Theorem A is a *veering triangulation* τ associated to the flow φ as introduced by Agol-Guéritaud ([1] and [12]) and explained by Landry, Minsky, Taylor [14]. The 2-skeleton of the veering triangulation $\tau^{(2)}$ is a branched surface which carries surfaces transverse to φ . In [20], Zung shows that any foliation $\mathcal{F}$ transverse to φ splits open to a lamination carried by $\tau^{(2)}$.

In light of this, we prove Theorem A by first constructing a branched submersion ψ according to the combinatorics of τ such that $\psi|_{\tau^{(2)}}$ is a branched submersion on every sector and $X(\psi)$ is a union of closed orbits of φ , including all the singular orbits. This is stated in Theorem 4.1. Then, by the work of Zung in [20], ψ will be a leafwise branched covering for every foliation transverse to φ .

Remark 1 (Transitive flows). We state Theorem A for **transitive** pseudo-Anosov flows to be consistent with other sources and results that we cite. These flows have a dense set of closed orbits. However, this is not much of a restriction: in [17], Mosher proves that all pseudo-Anosov flows are transitive in atoroidal 3-manifolds. Landry-Minsky-Taylor also discuss how this applies to compact 3-manifolds with boundary [15].

Many relevant examples. The question of when both structures exist in the same manifold—a pseudo-Anosov flow with a transverse foliation—is a key research question in the study of 3-manifolds. It is widely conjectured that for every taut foliation, there is an *almost* transverse pseudo-Anosov flow. This has been shown in the case of $\mathbb{R}$ -covered foliations by Calegari [4] and Fenley [9] and for foliations with one-sided branching by Calegari [5]. The Gabai-Mosher construction [18], which is complete in the work of Landry-Tsang [13], constructs a pseudo-Anosov flow almost transverse to a *finite-depth* taut foliation. From the other direction, Zung studies when there exists a taut foliation transverse to a pseudo-Anosov flow [20]. So despite not having a complete answer of when such situations occur, we have a lot of examples of both structures together.

1.1. Goals. Current research in the study of 3-manifolds often involves examining lower dimensional structures to reveal information about the relevant 3-manifold. We have already discussed (pseudo)-Anosov flows, foliations, and veering triangulations, but there is a menagerie of other structures: the Thurston norm ball, Heegaard splittings and Heegaard-Floer homology, universal circles, CaTherine wheels, and more. Many recent articles involve interactions between two or more of these or other structures (see [7], [8], [14] for example). One goal of this paper is to introduce leafwise branched coverings as another player and start to build connections with this network of structures.

Another goal is to give explicit examples of homotopies through branched coverings to develop visual intuition and understand them better. In Sections 3.1 and 3.4, we develop a

set of local moves which can be composed. In Sections 3.2 and 3.3, we discuss operations on these homotopies. And in Section 4, we use all these tools to construct the branched submersion for the main theorem. We hope that these explicit constructions will prove useful for further study of leafwise branched coverings and branched submersions.

Finally, the main construction inspired the definition of *taut cone fields* which come from a *totally vicious* Lorentzian metric. This property is well-studied in Lorentzian causality theory ([16] and [19]), but the connection with taut foliations has not been studied before, to the author's knowledge. We make the connection to introduce taut cone fields as another potential player in the study of 3-manifolds.

1.2. 3-manifolds fibering over S^1 : a first example. This paper is part of a broader research goal to understand explicit constructions of leafwise branched coverings. For example, we consider the case of a 3-manifold fibering over the circle.

Example 1.1 (3-manifold fibering over S^1). Suppose that M fibers over S^1 with a genus g surface fiber Σ . Let $f : \Sigma \rightarrow \Sigma$ be the monodromy of this fibration so that

$$M \cong (\Sigma \times [0, 1]) / (x, 1) \sim (f(x), 0).$$

Let $\mathcal{F}$ be the foliation by fibers on M given by $\{\Sigma \times \{t\}\}$. Suppose that there is a finite order homeomorphism $h : \Sigma \rightarrow \Sigma$ such that $f \circ h = h \circ f$ and the quotient $\Sigma / \langle h \rangle$ is homeomorphic to S^2 . Let $q_h : \Sigma \rightarrow S^2$ be the quotient map by h . Then f descends to a map $\bar{f} : S^2 \rightarrow S^2$ of the sphere with marked points at critical values of q_h . We then have the commutative diagram below.

$$\begin{array}{ccc} \Sigma & \xrightarrow{f} & \Sigma \\ q_h \downarrow & & \downarrow q_h \\ S^2 & \xrightarrow{\bar{f}} & S^2 \end{array}$$

Note that $\bar{f}$ is homotopic to Id after forgetting the critical values of q_h since $\text{MCG}(S^2)$ is trivial. Let $g_t : S^2 \rightarrow S^2$ be this homotopy so that $g_0 = \text{Id}$ and $g_1 = \bar{f}$. Then, define $\tilde{\psi} : \Sigma \times \mathbb{R} \rightarrow S^2$ with $\tilde{\psi}_t := \tilde{\psi}|_{\Sigma \times \{t\}}$ so that $\tilde{\psi}_t = g_t \circ q_h$. Notice that

$$\tilde{\psi}_1 = g_1 \circ q_h = \bar{f} \circ q_h = q_h \circ f = \tilde{\psi}_0 \circ f.$$

Since $\Sigma \times \mathbb{R}$ covers M , this condition implies that $\tilde{\psi}$ descends to a map $\psi : M \rightarrow S^2$. Thus, we have a leafwise branched covering on M . For example, h could be the hyperelliptic involution and f can be any hyperelliptic homeomorphism.

Berstein and Edmonds address a more general case of this example in Theorem 4.1 of [3]. They show that if $f : \Sigma \rightarrow \Sigma$ is any homeomorphism and $\psi_0 : \Sigma \rightarrow S^2$ is any branched covering of degree ≥ 3 , then there are homeomorphisms $g : \Sigma \rightarrow \Sigma$ and $\bar{g} : S^2 \rightarrow S^2$ with g isotopic to f such that $\bar{g} \circ \psi_0 = \psi_0 \circ g$. By isotoping $\bar{g}$ to Id and g to f , we have a homotopy $\tilde{\psi}_t$ through branched coverings between ψ_0 and $\psi_0 \circ f$. Again, this descends to a leafwise branched covering $\psi : M_f \rightarrow S^2$ on the mapping torus M_f .

1.3. Outline of paper. The goal of Section 2 is to pick out closed orbits of φ which will form the critical locus of the map ψ . We start by summarizing the construction of a veering triangulation τ from a pseudo-Anosov flow. We then recall key combinatorial structures associated to τ : the 2-skeleton $\tau^{(2)}$, the stable branched surface $\mathcal{B}^s$, the dual graph Γ , and the flow graph Φ . Then we apply work of Landry-Minsky-Taylor [14] to resolve vertices of Γ and split the graph into directed cycles that are all homotopic to closed orbits of the flow.

These closed orbits, along with a set coming from Φ and the singular orbits of φ , will form the critical locus of ψ .

In Section 3, we introduce homotopies through branched coverings as a way of constructing branched submersions. We define a handful of specific homotopies which are used in the construction for Theorem A and then consider a particularly useful class of branched coverings, Belyi branched maps.

Section 4 gives the construction of the branched submersion for Theorem A. We use the tools from Sections 2 and 3 in this construction. Then we explain how the resulting map is a leafwise branched covering for a transverse taut foliation and we see that the critical locus is a union of closed orbits of φ .

Inspired by necessary transversality properties in the main construction, in Section 5 we define taut cone fields. These cone fields come from a choice of a Lorentzian metric. We draw from ideas in Lorentzian causality theory including *totally vicious metrics* and *Lorentzian branched submersions* to adapt the definition of tautness.

Section 6 applies the construction in Section 4 to give an explicit branched submersion on the figure 8 knot complement which is a leafwise branched covering for the fibration with torus fiber.

Acknowledgements. I would like to thank Jonathan Zung for initially asking the motivating question for this paper and proposing the construction in Section 4. Jeffrey Norton introduced me to the work of Bernstein and Edmonds which proved to be useful in formalizing Section 3. I would especially like to thank my advisor, Danny Calegari, for his encouragement and support throughout this project. His insights were useful for the details of many proofs, and his speculations led to the definition of taut cone fields.

Statement on AI usage. Generative AI was used to make a handful of TikZ images and for a literature search. It was not used for any of the writing or mathematical content of this paper.

2. VEERING TRIANGULATIONS

The goal of this section is to produce directed cycles of the dual graph and the flow graph for a veering triangulation which are homotopic to closed orbits of the relevant flow. This is done in Propositions 2.10 and 2.11. These closed orbits will later form part of the critical locus for the branched submersion constructed in Section 4. We will start by introducing necessary background on pseudo-Anosov flows and veering triangulations.

2.1. Features of pseudo-Anosov flows. For a thorough treatment of pseudo-Anosov flows, we refer the reader to Mosher [17], Fenley-Mosher [10], and Agol-Tsang [2]. Heuristically, a **pseudo-Anosov flow** φ on M is a flow with specific expanding and contracting dynamics. This is witnessed by two invariant transverse singular two-dimensional foliations of M called the **unstable** and **stable foliations**. They intersect along flow lines of φ and orbits in the same leaf of the stable foliation are exponentially contracted by φ whereas orbits in the same unstable leaf are exponentially expanded by φ ([14], Section 4). Orbits of φ can be **closed/periodic**. We typically work with **transitive flows**, so the set of closed orbits is dense in the orbit space. In [17], Mosher proves that all pseudo-Anosov flows on an atoroidal 3-manifold are transitive, so this is not a major restriction. A **singular orbit** is a closed orbit at the intersection of singular leaves of the stable and unstable foliations. Figure 1 shows a foliated neighborhood of a singular and a nonsingular orbit. By definition, a pseudo-Anosov flow has only finitely many singular orbits.

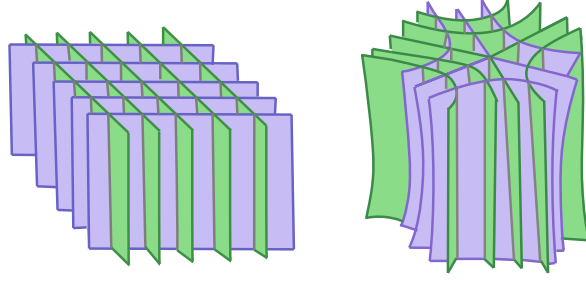


FIGURE 1. Orbits of a pseudo-Anosov flow are at intersections of the unstable and stable foliations. (Left) a neighborhood of a singular orbit and (right) a neighborhood of a nonsingular orbit.

Let $\tilde{\varphi}$ be the lifted flow on the universal cover $\tilde{M}$; the stable and unstable foliations also lift to $\tilde{M}$. Let $\mathcal{Q} := \tilde{M}/\tilde{\varphi}$ be the flow space of the lifted flow. In [10], Fenley-Mosher show that $\mathcal{Q}$ is a plane and the lifted stable and unstable foliations project to two singular transverse one-dimensional foliations on $\mathcal{Q}$ which we will call $\mathcal{F}^u$ and $\mathcal{F}^s$. Note that $\pi_1(M)$ acts on $\tilde{M}$ by deck transformations. This also gives an action on the lifted stable and unstable foliations. Then the action descends to $\mathcal{Q}$ and $\mathcal{F}^{u/s}$.

Given a finite collection of closed orbits X of the flow, we can perform a Fried blowup for each of them to obtain a **blown up flow** $\varphi^\#$ on the complement of a neighborhood of X . This process is described in Landry-Minsky-Taylor [15], Section 3. The new flow $\varphi^\#$ agrees with $\varphi|_{M \setminus X}$ on the interior of the manifold. For an orbit $\gamma \in X$, the corresponding torus boundary component has **stable and unstable prong curves** corresponding to the half leaves of the stable and unstable foliations containing γ . For a compact 3-manifold M with toral boundary, we say that φ is pseudo-Anosov if there is a Dehn filling of M so that φ is the blown up flow of a pseudo-Anosov flow on the filled manifold.

As described in [15] there is a minimum finite collection κ of periodic orbits of φ which *kills the perfect fits* of φ as described below. Let κ_S be the union of κ with the singular orbits and let $\tilde{\kappa}_S$ be the set of all lifts. Then we define $\mathcal{O} = \mathcal{Q} \setminus \tilde{\kappa}_S$. A **rectangle** in $\mathcal{O}$ is a map $R : (0, 1)^2 \rightarrow \mathcal{O}$ which is a homeomorphism onto its image so that the horizontal and vertical product foliations are mapped into $\mathcal{F}^s$ and $\mathcal{F}^u$ respectively. We say that κ *kills the perfect fits* of φ if every rectangle is contained in a unique **maximal rectangle** in $\mathcal{O}$ (see [14] page 22). Such a maximal rectangle will have a drilled (or removed) orbit on each of its four edges. See Figure 3.

2.2. A veering triangulation from a pseudo-Anosov flow. We now summarize the Agol-Guèrtaud construction of a *veering triangulation* for a pseudo-Anosov flow and important features of the tetrahedra within this triangulation. For a full explanation of this construction, see [14]. Let κ be a collection of closed orbits which kill the perfect fits for φ , let κ_S be the union of κ with the singular orbits of φ , and define $\mathring{M} := M \setminus \kappa_S$. A veering triangulation τ is a cellulation of $\mathring{M}$ by ideal tetrahedra satisfying extra combinatorial properties. A tetrahedron $\Delta \in \tau$ is called a **veering tetrahedron**. These properties are also summarized in [15].

- Each tetrahedron Δ has two faces cooriented outward, called the *top faces*, and two faces cooriented inward called the *bottom faces*.

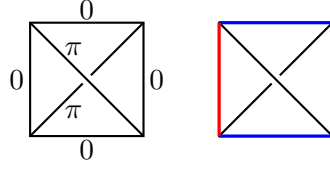


FIGURE 2. The local model of a veering tetrahedron.

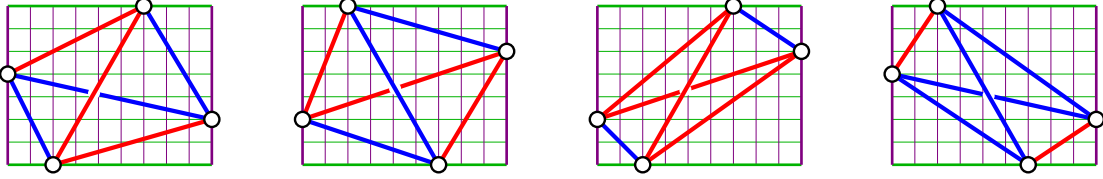


FIGURE 3. A maximal rectangle in $\mathcal{O}$ looks like one of these four options. The open circles represent drilled orbits in $\tilde{X}$. The two rectangles on the left are toggle rectangles and the two on the right are fan rectangles. The edges of the corresponding veering tetrahedra are drawn connecting the drilled orbits at their vertices.

- The edge between the top faces is assigned an angle of π , along with the edge between the bottom faces. These are called the *top edge* and the *bottom edge*, respectively.
- The other edges of Δ are called the equatorial edges and are all assigned an angle of 0.
- Each edge of the triangulation is assigned a *color* or *veer*— **Red/Right** or **bLue/Left** so that every Δ is locally modeled by Figure 2.
- A tetrahedron is *toggle* if its top and bottom edge have opposite veers and it is *fan* if they have the same veer.

Agol and Guéritaud proceed by building an ideal triangulation $\tilde{\tau}$ of $\tilde{M} \setminus \tilde{\kappa}_S$ with a veering tetrahedron for each maximal rectangle in $\mathcal{O}$. Figure 3 depicts the four combinatorial types of maximal rectangles and the projection of the corresponding tetrahedron Δ to $\mathcal{O}$. The vertices of Δ lie at drilled orbits and the edges of Δ are diagonals of the rectangle connecting the drilled orbits. Edges with positive slope are colored red while those with negative slope are colored blue. The coorientation of faces is given by the flow direction. Note that the top edge of Δ connects the two horizontal boundaries of the rectangle and the bottom edge connects two vertical boundaries.

After building $\tilde{\tau}$ from the maximal rectangles, Agol-Guéritaud show there are finitely many orbits of maximal rectangles under the action of $\pi_1(M)$. This defines a finite triangulation $\tau := \tilde{\tau}/\pi_1(M)$ of $\mathring{M}$.

Theorem 2.1 (Agol-Guéritaud, Landry-Minsky-Taylor [15]). *There is a veering triangulation τ of $M \setminus \kappa_S$ such that $\tau^{(2)}$ is a smooth co-oriented branched surface positively transverse to the flow lines of φ .*

2.3. Dual graph and flow graph. Veering triangulations have proved extremely useful for studying pseudo-Anosov flows because they encode the dynamics of the associated flow combinatorially. We now introduce a handful of structures— the stable branched surface,

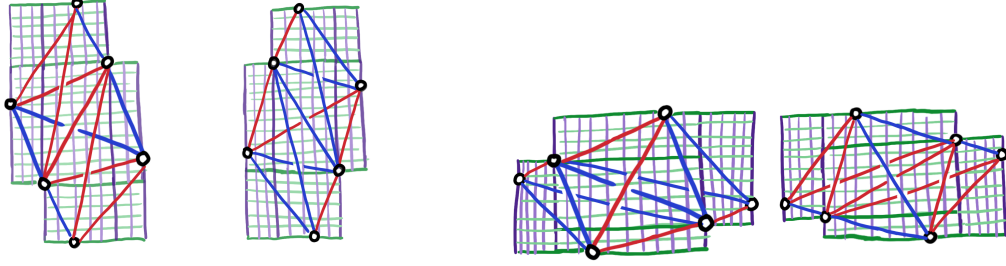


FIGURE 4. The tetrahedra glued to the top faces of a tetrahedron Δ spiral clockwise if the top edge is red and counterclockwise if the top edge is blue (left). The tetrahedra glued below Δ spiral opposite directions depending on the veer of the bottom edge (right).

dual graph, and flow graph—to see this in action. In what follows, let τ be a veering triangulation on $\tilde{M} = M \setminus \kappa_S$ for a killing set κ for the pseudo-Anosov flow φ .

Partial order on rectangles. Note that $\tilde{M}$ has a coorientation given by the flow direction. We imagine that some tetrahedra in $\tilde{\tau}$ lie above others. To formalize this notion, we define a partial order on rectangles in $\mathcal{O}$ which is compatible with this coorientation, following the convention in [14].

If the interiors of two rectangles R_1 and R_2 intersect, we say that they are ordered. For two ordered rectangles, $R_1 < R_2$ if R_2 is at least as tall as R_1 and R_1 is at least as wide as R_2 , and $R_1 \neq R_2$. If this holds, then we say R_2 **lies above** R_1 . Heuristically, rectangles get skinnier and taller as we move in the forward direction according to the flow.

This partial order is compatible with the flow direction on $\tilde{M}$ in the following sense. Suppose that $g \in \pi_1(M)$ fixes a flow line $p \in \mathcal{O}$ and translates in the positive direction. Then for any maximal rectangle R containing p , $g(R)$ lies above R ([14], Lemma 4.2).

Dual graph. Consider now the tetrahedron Δ for a maximal rectangle R . Each face of Δ corresponds to a rectangle with 3 drilled orbits on its boundary for the three vertices of the face. Each face rectangle is contained in a unique maximal rectangle. The maximal rectangles for the top faces will lie above R , and those for the bottom faces will lie below R .

We observe that if Δ has a red top edge, the tetrahedra glued to its top faces spiral clockwise in the flow direction. If Δ has a blue top edge, the tetrahedra glued above spiral counterclockwise in the flow direction. This is shown in Figure 4. In the same way, the veer of the bottom edge of Δ determines the spiral direction for the tetrahedra glued to the bottom faces.

Definition 2.2. The *stable branched surface* $\mathcal{B}^s$ is the dual 2-complex to τ with a local smoothing condition. Inside a tetrahedron Δ of τ , $\mathcal{B}^s$ is smoothed according to Figure 5¹ depending on the veer/color of the top edge of Δ .

The *dual graph* Γ is the branch locus (1-skeleton) of $\mathcal{B}^s$. It is a 4-valent directed graph. The vertices represent tetrahedra in τ and are connected by a directed edge if they share a face with the edge oriented by the flow direction. From the earlier discussion, each vertex

¹Thanks to Landry-Minsky-Taylor for the picture of the stable branched surface in Figure 5.

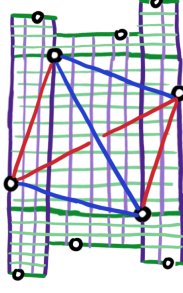


FIGURE 6. Three maximal rectangles containing an edge of Δ lie above Δ .

has two incoming edges and two outgoing edges. For a tetrahedron Δ , we use v_Δ to denote the vertex of Γ representing Δ .

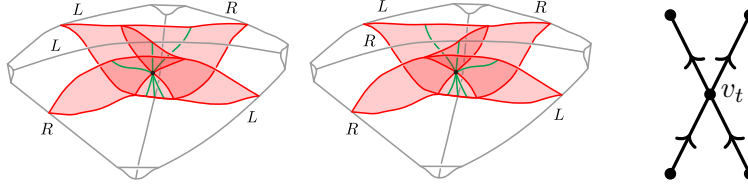


FIGURE 5. The smoothing of $\mathcal{B}^s$ inside tetrahedra with left veering top edge and right veering top edge. $\mathcal{B}^s$ is in red, edges of Γ are bolded red lines, and edges of $\iota(\Phi)$ are green. A piece of the dual graph Γ is shown on the right.

At a vertex v_Δ of the dual graph Γ , there are two ways to pair the incoming edges with outgoing edges (or the bottom faces of Δ with the top faces of Δ). We call such a choice a *resolution* of Γ at v_Δ . One choice follows the smoothly embedded branch locus of $\mathcal{B}^s$, and the other choice has corners with respect to this smoothing. A directed path in Γ of length 2 has one turn: this is called a *branch turn* if it is smoothly embedded in the branch locus and an *anti-branching* (AB) turn if not. A directed path/ray/cycle in Γ is *branching* if it consists of only branch turns and *anti-branching* (AB) if it consists of only AB turns.

Flow graph. Now consider the tetrahedra above Δ which share an edge with Δ . There are three such tetrahedra and their corresponding maximal rectangles are shown in Figure 6. Note that they include the top edge and the two side edges with opposite veer as the top edge. The flow graph will encode the relationship between these tetrahedra.

Definition 2.3. The *flow graph* Φ has vertices representing edges in τ . For each tetrahedron, there is a directed edge in Φ from the bottom edge to the top edge, and from the bottom edge to the two side edges with opposite veer as the top edge.

For each edge e in τ , there is a unique tetrahedron in τ with e as the bottom edge; we call this tetrahedron $\Delta(e)$. Let $\iota : \Phi \rightarrow \mathcal{B}^s$ be an embedding of the flow graph which sends a vertex e to the vertex $v_{\Delta(e)}$ of Γ .

Figure 5 demonstrates the interaction of $\mathcal{B}^s$, Γ , and $\iota(\Phi)$ inside the manifold. In [14], Landry-Minsky-Taylor show that directed cycles of the dual graph and flow graph encode closed orbits of the flow φ . They also describe exactly when this encoding is unique, although we will not need uniqueness for our purposes.

In Proposition 3.15, they show that all directed cycles of Γ are homotopic to a flow cycle or an odd length AB-cycle. During the proof, they conclude that if γ is homotopic to an odd AB-cycle, then it is homotopic to the core of an embedded Möbius band in a leaf of the stable foliation on M . This core curve is homotopic to a closed orbit of the flow. Then, in Theorem 6.1, the authors show that every closed directed cycle of Φ is homotopic to a closed orbit of the flow or an unstable prong curve of $\varphi^\#$. Landry-Minsky-Taylor also investigate when this homotopy can be made always transverse to $\tau^{(2)}$ in Proposition 6.7. We summarize these results in the following theorem.

Theorem 2.4 (From Proposition 3.15, Theorem 6.1, and Proposition 6.7 in [14]). *Every closed directed cycle in the dual graph Γ or the flow graph Φ of τ is homotopic to a closed orbit of the flow φ or an unstable prong curve. Further, for a closed directed cycle of Φ and a non-branch curve of Γ , the homotopy can be made always transverse to $\tau^{(2)}$.*

The idea of the proof is that a directed cycle in Φ (or Γ) lifts to a bi-infinite path in the universal cover $\tilde{\Phi}$ (or $\tilde{\Gamma}$) in $\tilde{M}$. The corresponding sequence of maximal rectangles in $\mathcal{O}$ get taller and skinnier in the forward flow direction, whereas they get shorter and wider in the backward flow direction. Then, applying the contracting map theorem in both horizontal and vertical directions picks out a single point in the infinite intersection. This point in $\mathcal{O}$ will be a flow line invariant under the action of some element in $\pi_1(M)$, and this a closed orbit of the flow.

2.4. Resolving into directed cycles. The goal of this section is to pick out a collection of directed cycles of Γ and Φ which are homotopic to genuine closed orbits of φ , and not unstable prong curves. For Γ , we do this by finding an appropriate resolution. In this section, we will use Δ to refer to both a tetrahedron in $\tilde{\tau}$ and a maximal rectangle in $\mathcal{O}$.

Local directions for a tetrahedron. Fix a maximal rectangle Δ in $\mathcal{O}$ and identify it with $(0, 1) \times (0, 1)$ so that the horizontal foliation is along stable leaves and the vertical foliation is along unstable leaves. Such an identification assigns directions to Δ : up, down, left, and right. There are two orientation-preserving choices for this identification which differ by a 180° rotation. We will use these local directions to label edges in a piece of $\tilde{\Gamma}$.

If Δ' is any maximal rectangle comparable with Δ in the partial order, then Δ' is given directions consistent with those on Δ . There are two rectangles above Δ which share a face with Δ , we will call them $L_\Gamma(\Delta)$ and $R_\Gamma(\Delta)$ for one is to the left and one is to the right. Below Δ , there are also two rectangles which share a face with Δ ; these will be called $U_\Gamma(\Delta)$ and $D_\Gamma(\Delta)$ since one is “up” and one is “down”.

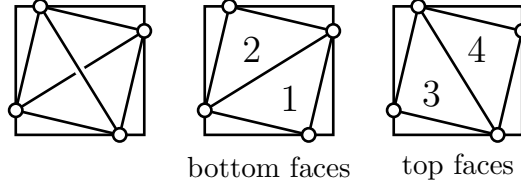
Observation 2.5. Fix directions on Δ and let Δ' be any vertex in $\tilde{\Gamma}$ comparable to Δ in the partial order. Then the outgoing edges from Δ' are of the form

$$\Delta' \rightarrow R_\Gamma(\Delta') \quad \text{and} \quad \Delta' \rightarrow L_\Gamma(\Delta').$$

The incoming edges to Δ' are of the form

$$U_\Gamma(\Delta') \rightarrow \Delta' \quad \text{and} \quad D_\Gamma(\Delta') \rightarrow \Delta'.$$

So for every edge in $\tilde{\Gamma}$ with endpoints comparable to Δ , we can label it with one of R or L and one of U or D . The R/L indicate which form the edge takes in the forward direction, and the U/D indicates which form it takes in the backward direction.

FIGURE 7. Label the faces of Δ with 1, 2, 3, 4.

Observation 2.6. A turn in $\tilde{\Gamma}$ (a path of length 2) with Δ as a middle vertex will have a labeling in the forward direction by R/L and a labeling in the backward direction by U/D as described in Observation 2.5. Each of these labelings has four possible forms:

forward labeling: $L \rightsquigarrow L$, $L \rightsquigarrow R$, $R \rightsquigarrow L$, $R \rightsquigarrow R$.

backward labeling: $U \leftrightsquigarrow U$, $U \leftrightsquigarrow D$, $D \leftrightsquigarrow U$, $D \leftrightsquigarrow D$.

A resolution of the vertex v_Δ in Γ will result in two directed 2-paths. There are two possible cases for the forward direction labeling of these turns: either (i) $R \rightsquigarrow R$ and $L \rightsquigarrow L$, or (ii) $R \rightsquigarrow L$ and $L \rightsquigarrow R$. We have analogous possibilities for their backward direction.

Lemma 2.7. *Suppose that the branching resolution of v_Δ is made. In the backward direction, the resulting turns will be labeled*

$$U \leftrightsquigarrow U \quad \text{and} \quad D \leftrightsquigarrow D.$$

If v_Δ is a fan tetrahedron, in the forward direction, the resulting turns will follow situation (i). If v_Δ is a toggle tetrahedron, the result is situation (ii).

Proof. Let the faces of Δ be labeled as in Figure 7. From studying the picture of $\mathcal{B}^s$ in Figure 5, we determine that if the top edge of Δ is red, the branching resolution pairs faces $1 \leftrightarrow 3$ and $2 \leftrightarrow 4$. If the top edge is blue, the pairing is opposite: $1 \leftrightarrow 4$ and $2 \leftrightarrow 3$.

Figure 4 shows tetrahedra glued along faces of a toggle tetrahedron. From the face pairing for the branching resolution, we determine that the resulting 2-paths will be labeled $U \leftrightsquigarrow U$ and $D \leftrightsquigarrow D$ in the backward direction and $R \rightsquigarrow L$ and $L \rightsquigarrow R$ in the forward direction.

Then we examine Figure 8 for the case of a fan tetrahedron. Again, we consider the face pairing from the branching resolution and determine that the resulting 2-paths will be labeled $U \leftrightsquigarrow U$ and $D \leftrightsquigarrow D$ in the backward direction and $L \rightsquigarrow L$ and $R \rightsquigarrow R$ in the forward direction. □

We can make a similar convention for the flow graph. Every vertex of $\tilde{\Phi}$ has three directed outgoing edges. By assigning directions to a fixed rectangle Δ , we can label the terminal endpoints of these outgoing edges by $L_\Phi(\Delta)$, $M_\Phi(\Delta)$, $R_\Phi(\Delta)$ for the left, middle, and right rectangles above Δ . In the same way as above, any edge in $\tilde{\Phi}$ with endpoints comparable to Δ can be labeled with a letter L , M , or R in the forward direction. We will not need backward labeling for the edges of the flow graph.

Avoiding singular orbits. We will now use local directions on a fixed tetrahedron to choose closed directed cycles of Γ and Φ which are not homotopic to unstable prong curves.

Lemma 2.8. *Let $\dots \Delta_{-1} \rightarrow \Delta_0 \rightarrow \Delta_1 \rightarrow \dots$ be a bi-infinite directed path of $\tilde{\Phi}$ such that there is some k with $\Delta_{k+1} = M(\Delta_k)$. Then $\bigcap \Delta_i$ contains no singular orbit of $\mathcal{O}$.*

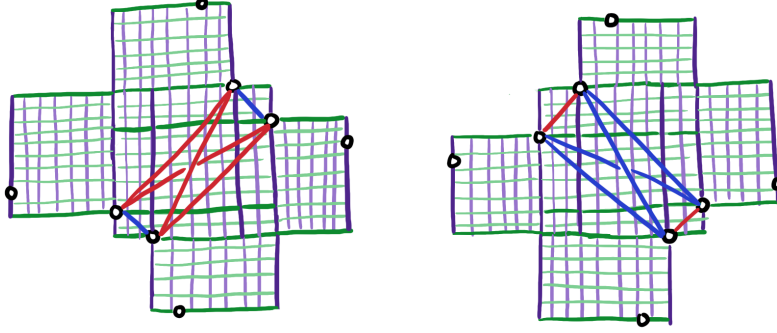


FIGURE 8. The maximal rectangles corresponding to tetrahedra above and below Δ in the dual graph when Δ is fan.

Proof. There are three possibilities for a turn with Δ_k as a middle vertex. They are $R \rightsquigarrow M$, $L \rightsquigarrow M$, or $M \rightsquigarrow M$. In all cases, we look at $\Delta_{k-1} \cap \Delta_k \cap \Delta_{k+1}$ and we see that the intersection contains no singular orbits. $\square$

Lemma 2.9. *Let $\dots \Delta_{-1} \rightarrow \Delta_0 \rightarrow \Delta_1 \rightarrow \dots$ be a bi-infinite directed path of $\tilde{\Gamma}$ such the sequence of labels R/L and U/D for the edges is periodic. Suppose that there is at least one of L, R, U , and D in the sequence. Then $\bigcap \Delta_i$ contains no singular orbit of $\mathcal{O}$.*

Proof. Consider the rectangle Δ_0 . We need only to argue that none of the singular orbits on its boundary can be in the intersection $\bigcap \Delta_i$.

We start by considering $\Delta_{-1} \cap \Delta_0 \cap \Delta_1$. This intersection will be a rectangle with two singular orbits on the boundary. Consider the case where

$$\Delta_{-1} = D_\Gamma(\Delta_0) \quad \text{and} \quad R_\Gamma(\Delta_0) = \Delta_1.$$

Then $\Delta_{-1} \cap \Delta_0 \cap \Delta_1$ is a rectangle with a singular orbit p_1 on the right boundary and a singular orbit on the bottom boundary, p_2 . We will give an argument which excludes both of these from the intersection $\bigcap \Delta_i$. The argument for the three other cases (choosing D/U and R/L) follows similarly.

Note that every tetrahedron Δ_i above Δ_0 in the sequence either has p_1 on its right boundary or does not contain p_1 in $\overline{\Delta_i}$. Since the sequence of edge labelings is periodic and contains some L , there is a $k > 0$ such that $\Delta_{k+1} = L_\Gamma(\Delta_k)$. The right boundary of Δ_{k+1} is to the left of the right boundary of Δ_k . Thus, $p_1 \notin \overline{\Delta_{k+1}}$.

Now we address p_2 . Every tetrahedron Δ_i below Δ_0 in the sequence either has p_2 on its bottom boundary or does not contain p_2 in $\overline{\Delta_i}$. Again, the sequence of edge labelings is periodic and contains some U , so there is a $\ell < 0$ such that $\Delta_{\ell-1} = U_\Gamma(\Delta_\ell)$. The bottom boundary of $\Delta_{\ell-1}$ is strictly above the bottom boundary of Δ_ℓ . Thus, $p_2 \notin \overline{\Delta_{\ell-1}}$.

This implies that $\Delta_{\ell-1} \cap \Delta_{-1} \cap \Delta_0 \cap \Delta_1 \cap \Delta_{k+1}$ contains no singular orbits. Thus, there are no singular orbits in the infinite intersection. $\square$

Proposition 2.10. *Let $d_1, \dots, d_m$ be the collection of directed cycles of the flow graph Φ consisting of edges $\Delta \rightarrow M(\Delta)$. Then each d_i is transversely homotopic to a closed orbit of the flow.*

Proof. From Theorem 6.1 in [14], we have that each d_i is transversely homotopic to some closed orbit of the flow or some unstable prong curve. By Lemma 2.8, it cannot be an unstable prong curve.

□

Proposition 2.11. *There is a decomposition of Γ into directed cycles $c_1, \dots, c_k$ with each c_i transversely homotopic to a closed orbit of the flow φ .*

Proof. The dual graph Γ is a finite directed graph where each vertex represents either a toggle or a fan tetrahedron. Note that any directed cycle consisting of only toggle tetrahedra must have an even number of vertices. This follows because when two toggle tetrahedra share a face, they must have opposite veers for their top edges.

Now we assign a branching (B) or anti-branching (AB) resolution to the vertices of Γ . If a vertex represents a fan tetrahedron, give it an anti-branching resolution. Consider the induced subgraph of Γ on the vertices representing toggle tetrahedra. Then assign B or AB to these vertices so that no two vertices sharing an edge have the same assignment. We can do this because this induced subgraph has only even cycles. Then give the tetrahedra the corresponding resolutions.

We have now resolved all vertices of Γ ; let $c_1, \dots, c_k$ be the resulting directed cycles. By construction, each of these either (i) passes through at least one fan tetrahedron or (ii) passes through two toggle tetrahedra, one of each resolution. Thus, by Lemma 2.7, each c_i will have at least one of R , L , U , and D in its edge labeling sequence. We apply Lemma 2.9 to their lifts $\tilde{c}_i$ to conclude that none of c_i are homotopic to unstable prong curves. Additionally, we notice that no c_i is homotopic a branch curve of Γ , and so the homotopy can be made transverse to $\tau^{(2)}$. Hence, applying the results from [14] summarized in Theorem 2.4, we have that each c_i is transversely homotopic to a closed orbit of φ .

□

3. BRANCHED COVERINGS

The goal of this section is to construct maps which will serve as building blocks for the main construction in Section 4. These are defined in Section 3.1 and Section 3.4. In Sections 3.1 and 3.3, we introduce homotopies through branched coverings and branched submersions which offer a new perspective on leafwise branched coverings. Section 3.4 defines Belyi maps which are a key tool for Section 4. Much of the language and proof techniques for this section are from “On the Construction of Branched Coverings of Low-dimensional Manifolds” by Bernstein and Edmonds [3].

A **branched covering** between topological manifolds $\psi : M^n \rightarrow N^n$ (without boundary) is an immersion except at a **singular set** $X(\psi)$ of M which is a subpolyhedron of codimension 2. Around this singular set, ψ acts as a quotient by a finite group of homeomorphisms. If M and N are compact, ψ is finite-to-one, and we define the **degree of ψ** to be $\max\{\psi^{-1}(y) \mid y \in N\}$.

For a branched covering between surfaces $\psi : S \rightarrow T$, the singular set is a discrete set of **critical points** $p_1, \dots, p_n$ such that the restricted map $\psi : S \setminus \{\psi^{-1}(\psi(p_i))\} \rightarrow T \setminus \{\psi(p_i)\}$ is a covering map. Around each point x in S , ψ can be locally modeled in complex coordinates by $z \mapsto z^k$ for $k \geq 1$ (with $z(x) = 0$ in the coordinate system). The **degree at x** is k in the above definition; this is denoted $\deg(\psi; x)$. A critical point of ψ has degree ≥ 2 and the **order** of the critical point is $k - 1$. The image of a critical point is a **critical value**.

More generally, we will consider maps $\psi : S \rightarrow T$ between surfaces with boundary. Again, let ψ be an immersion outside a discrete set of critical points $p_1, \dots, p_n$ in $\text{int}(S)$ at which ψ is locally modeled by $z \mapsto z^k$ for $k \geq 2$. If ψ is a proper map, we will still call it a branched covering since it is a covering map outside the critical points. Otherwise, ψ will be called a **branched immersion**. The definitions for degree, order, and critical values are still the same. In the case of a branched immersion, there may not be a well-defined global degree.

3.1. Homotopies through branched coverings. Two branched coverings $\psi_0, \psi_1 : M^n \rightarrow N^n$ are **b-homotopic** if there is a homotopy $\Psi_t, t \in [0, 1]$ between ψ_0 and ψ_1 with each Ψ_t a branched covering. The map $\Psi : M \times [0, 1] \rightarrow N$ is called a **b-homotopy**. The degree of a b-homotopy is the degree of Ψ_t for any t (note that the degree is constant).

For branched coverings of surfaces with boundary, we require a b-homotopy to be constant on the boundary. We can make an analogous definition for b-homotopies of branched immersions. Most of the time, these will also be homotopies relative to the boundary. In this last situation, the degree is not defined.

Let $\Psi : S \times [0, 1] \rightarrow T$ be a b-homotopy for surfaces S and T . Then define $X(\Psi) := \bigcup_{t \in [0, 1]} X(\Psi_t)$ to be the **singular set**. We notice that (i) Ψ is a submersion outside of $X(\Psi)$; it is given locally by a continuous family of immersions. Also, by construction, (ii) in a neighborhood of the singular set, Ψ is locally a continuous family of branched immersions. A map from a 3-manifold to a surface with a 1-dimensional singular set and satisfying conditions (i) and (ii) will be called a **branched submersion**. Leafwise branched coverings are examples of branched submersions.

Also, we notice that Ψ factors as a composition of a 3-manifold branched covering and a projection. Let $\hat{\Psi} : S \times [0, 1] \rightarrow T \times [0, 1]$ be given by $\hat{\Psi}(x, t) = (\Psi_t(x), t)$ and $\pi : T \times [0, 1] \rightarrow T$ be projection onto the first factor. Then $\Psi = \pi \circ \hat{\Psi}$ and $X(\Psi)$ is also the singular set of $\hat{\Psi}$. Locally, all branched submersions take on this form.

Example 3.1 (b-homotopies on a disk). Any continuous family of degree d rational maps $\psi_t : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ is an example of a b-homotopy. In fact, these locally parameterize all b-homotopies in the same way that rational maps locally parameterize branched coverings. In particular, consider the following specific examples.

- (1) **Splitting off a critical point:** $f_t(z) = z^{k-1}(z - t)$ for $t \in [0, 1]$. At $t = 0$, f_0 has one critical point of order $k - 1$. At every non-zero t , f_t has one critical point of order $k - 2$ and one of order 1.
- (2) **Do-si-do of critical points:** $g_t(z) = z^3 + e^{2\pi it}z$ for $t \in [0, 1]$. Note that $g_0 = g_1$ and g_t has two critical points of order 1 for all t . Throughout the homotopy, they change places.
- (3) **Do-si-do around an extra point:** $h_t = z^2(z^2 - e^{2\pi it})$ for $t \in [0, 1]$. For every t , h_t has one critical point of order 1 at $z = 0$ and two other critical points of order 1 which perform a half twist during the homotopy.

We modify each of these examples to build three b-homotopies $D^2 \times [0, 1] \rightarrow D^2$ with specified critical locus. Consider f_t as defined above. Let $\gamma \subset \hat{\mathbb{C}}$ be the circle of radius 2 centered at 0. Note that $f_t^{-1}(\gamma)$ is a Jordan curve for all t and the map

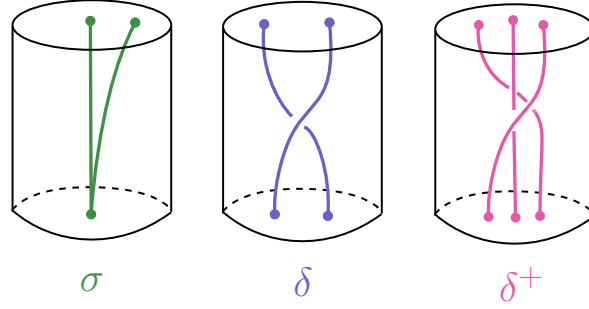
$$f_t : f_t^{-1}(\gamma) \rightarrow \gamma$$

is a k -fold covering map on the circle. Both γ and $f_t^{-1}(\gamma)$ bound a disk in $\mathbb{C}$, let them be called D and D_t respectively. Then identify $\bigcup D_t$ with $D \times [0, 1]$ with a fiberwise homeomorphism.

Then, up to a homeomorphism, f restricts to a branched submersion $f : D \times [0, 1] \rightarrow D$ such that $f|_{\partial D \times \{t\}}$ is a k -fold covering map and constant in t .

We can repeat this same construction for g and h as defined above. We obtain three b-homotopies $D \times [0, 1] \rightarrow D$ which we will call σ , δ , and δ^+ , coming from f , g , and h , respectively, shown in Figure 9.

3.2. Operations on b-homotopies.

FIGURE 9. Three b-homotopies $D^2 \times [0, 1] \rightarrow D^2$

- If $\psi : S \rightarrow T$ is a branched immersion, then we can **suspend** ψ to make a constant b-homotopy. This map is defined as $\hat{\psi} : S \times [0, 1] \rightarrow T$, $\hat{\psi}(x, t) = \psi(x)$.
- We define the **reflection** of a b-homotopy $\Psi : S \times [0, 1] \rightarrow T$ to be the map

$$\bar{\Psi} : S \times [0, 1] \rightarrow T, \quad \bar{\Psi}(x, t) := \Psi(x, 1 - t).$$

- Given two b-homotopies $\Psi, \Psi' : S \times [0, 1] \rightarrow T$ between the same surfaces, if $\Psi_1 = \Psi'_1$, we can combine them to make a single b-homotopy. The **stack** of Ψ and Ψ' is defined as

$$\Psi \cdot \Psi' : S \times [0, 1] \rightarrow T, \quad \Psi \cdot \Psi'(x, t) = \begin{cases} \Psi(x, 2t) & 0 \leq t \leq \frac{1}{2} \\ \Psi'(x, 2t - 1) & \frac{1}{2} \leq t \leq 1 \end{cases}.$$

Construction 3.2 (Inserting a b-homotopy in a disk). Suppose that $\Psi : S \times [0, 1] \rightarrow T$ is a b-homotopy for general surfaces and there is a continuous family of disks $D_t \subset S$ for which $\Psi|_{D_t \times \{t\}}$ is a branched covering onto an image disk in T . Then $\Psi|_{\partial D_t \times \{t\}}$ is a k -fold covering map on the circle for some k .

Let $H : D^2 \times [0, 1] \rightarrow D^2$ be any degree k b-homotopy. Then choose a homeomorphism identifying $\partial D^2 \times \{t\}$ with ∂D_t so that H and Ψ are equal on ∂D_t for all t .

Then we define a new map $\Psi \natural H$ by replacing $\Psi|_{\bigcup D_t}$ with H and gluing with this boundary homeomorphism. We call this an **insertion of H** .

3.3. Homotoping the singular set. We now describe ways to modify the singular set of a branched covering and a b-homotopy locally in order to achieve some desired arrangement.

In [3], Bernstein-Edmonds show that every branched covering $\psi : D^2 \rightarrow D^2$ is b-homotopic rel ∂D boundary to a simple branched covering (Lemma 3.2). In their proof, they actually show that every degree d branched covering $D^2 \rightarrow D^2$ is b-homotopic to the map $z \mapsto z^d$ on $\mathbb{D}$ in complex coordinates. This implies that every degree d branched covering from the disk to itself is b-homotopic. We can apply this proof technique to obtain an analogous result for b-homotopies.

Lemma 3.3. *Let $\Psi, \Psi' : D^2 \times [0, 1] \rightarrow D^2$ both be degree d b-homotopies. Then they are homotopic through b-homotopies rel $\partial D^2 \times [0, 1]$.*

Proof. Since Ψ is degree d then $\Psi|_{\partial D^2 \times \{t\}}$ is a d -fold covering map on the circle. By definition, $\Psi|_{\partial D^2 \times \{t\}}$ is constant in t . Up to homeomorphism, we may assume $\Psi|_{\partial D^2 \times [0, 1]} = \Psi'|_{\partial D^2 \times [0, 1]}$.

Let $f_d : D^2 \rightarrow D^2$ be the branched covering which is the cone on $\Psi|_{\partial D^2 \times \{0\}}$. In other words, f_d is equivalent to the map $z \mapsto z^d$ on the unit disk $\mathbb{D}$. Let $F_d : D^2 \times [0, 1] \rightarrow D^2$ be

the suspension of b-homotopy f_d . Then both Ψ and Ψ' are homotopic through b-homotopies to F_d by an Alexander trick applied simultaneously to each $D^2 \times \{t\}$. $\square$

Lemma 3.4. *Let $\Psi, \Psi' : D^2 \times [0, 1] \rightarrow D^2$ be b-homotopies such that $\Psi_0 = \Psi'_0$ and $\Psi_1 = \Psi'_1$. Then Ψ and Ψ' are homotopic through b-homotopies rel $\partial(D^2 \times [0, 1])$.*

Proof. Note that Ψ and Ψ' must have the same degree because of the condition on their boundary maps. We apply Lemma 3.3 to build a homotopy through b-homotopies between $\Psi|_{D^2 \times [\varepsilon, 1-\varepsilon]}$ and $\Psi'|_{D^2 \times [\varepsilon, 1-\varepsilon]}$. Let $G_t : D^2 \times [\varepsilon, 1-\varepsilon] \rightarrow D^2$ be this homotopy.

For each t , branched coverings Ψ_0 and $G_t|_{D^2 \times \{\varepsilon\}}$ are b-homotopic and the same for Ψ_1 and $G_t|_{D^2 \times \{1-\varepsilon\}}$. Let

$$K_t : D^2 \times [0, \varepsilon] \rightarrow D^2 \quad \text{and} \quad L_t : D^2 \times [1-\varepsilon, 1] \rightarrow D^2$$

be these respective b-homotopies and note that we can change them continuously with respect to t .

Then, stack $L_t \cdot G_t \cdot K_t =: H_t$ to get 1-parameter family of b-homotopies with $H_t|_{D^2 \times \{i\}} = \Psi_i$ for $i = 0, 1$. Then $H_0 = \Psi$ and H_1 is Ψ' with a possible slight perturbation. $\square$

Corollary 3.5 (of Lemma 3.3). *If two b-homotopies Ψ and Ψ' differ by an insertion in a disk according to Construction 3.2, then they are homotopic through b-homotopies.*

Two branched coverings $\psi_0, \psi_1 : M^n \rightarrow N^n$ are **equivalent** if there are homeomorphisms $f : M \rightarrow M$ and $g : N \rightarrow N$ such that $g \circ \psi_0 = \psi_1 \circ f$. If both g and f are isotopic to the identity, this equivalence gives a b-homotopy between ψ_0 and ψ_1 where the singular set is a 1-manifold. In the case that both M and N are either the disk or S^2 , all homeomorphisms are isotopic to the identity, forgetting marked points. We apply this observation for the following results.

One way to check for equivalence of branched coverings is to use **Hurwitz systems** (or monodromy representations). Let $\psi : S \rightarrow T$ be a branched cover of degree d . Let $B_\psi \subset T$ be its critical values and choose a basepoint $* \in T$. Then, upon choosing a labeling of $\psi^{-1}(*)$ with d letters $\{1, \dots, d\}$, we have a representation of the fundamental group of $T - B_\psi$ to the symmetric group.

$$\rho(\psi) : \pi_1(T - B_\psi, *) \rightarrow S_d$$

For each based loop in $T - B_\psi$, lift it to a sequence of arcs under the covering map to determine a permutation of $\psi^{-1}(*)$. This representation $\rho(\psi)$ is determined up to inner automorphism of S_n .

In [3] (2.1), Bernstein-Edmonds state the Hurwitz Classification Theorem which says that two branched coverings $\psi, \psi' : S \rightarrow T$ are equivalent if and only if there is a homeomorphism $h : (T, B_\psi, *) \rightarrow (T, B_{\psi'}, *)$ such that $\rho(\psi') = \rho(\psi)h_*$.

Lemma 3.6. *Any two branched coverings $\psi_0, \psi_1 : D^2 \rightarrow D^2$ with one critical point of degree n and one critical point of degree k and distinct critical values are equivalent.*

Proof. First, we note that the Riemann-Hurwitz formula implies that ψ_0 and ψ_1 have the same degree. Indeed, for a branched cover $\psi : S \rightarrow T$, we have

$$\chi(S) = \deg(\psi) \cdot \chi(T) - \sum (\deg(\psi; x) - 1).$$

Since ψ_0 and ψ_1 have the same numbers and degrees of critical points, they have the same total degree $d = n + k - 1$.

Next, we consider the possible Hurwitz systems for ψ_0 and ψ_1 . Note that B_{ψ_0} consists of two points b_1 and b_2 which are images of critical points with degree n and k , respectively.

Pick a basepoint $*$ in $D^2 - B_{\psi_0}$ and a labeling of $\psi_0^{-1}(*)$. Let α_1 and α_2 be based loops encircling b_1 and b_2 , respectively. Then $\sigma_1 := \rho(\psi_0)(\alpha_1)$ is an n -cycle and $\sigma_2 := \rho(\psi_0)(\alpha_2)$ is a k -cycle, and (up to a possible relabeling)

$$\sigma_1 \cdot \sigma_2 = (1 \ 2 \ 3 \ \dots \ d).$$

We claim that all such factorizations $\sigma_1 \cdot \sigma_2$ differ by an inner automorphism of S_d .

Note that σ_1 fixes $k - 1$ letters and σ_2 is of length k . Thus, since $\sigma_1 \cdot \sigma_2$ fixes no letters, there is only one letter appearing in both σ_1 and σ_2 . The action of σ_1 on the other $n - 1$ letters is prescribed, as well as the action of σ_2 on the $k - 1$ letters fixed by σ_1 . Thus, we must have

$$\sigma_1 = (\delta_1 \ \delta_2 \ \dots \ \delta_n) \quad \sigma_2 = (\delta_1 \ \delta_{n+1} \ \dots \ \delta_{n+k-1})$$

where $\delta_i + 1 = \delta_{i+1}$ with indices taken modulo $n + k - 1$. We then see that up to relabeling, all choices are equivalent. We can then apply the Hurwitz Classification Theorem to conclude that ψ_0 and ψ_1 are equivalent. $\square$

Corollary 3.7. *Let $\Psi : D^2 \times [0, 1] \rightarrow D^2$ be a b-homotopy such that $\Psi_0, \Psi_1 : D^2 \rightarrow D^2$ each have one critical point of degree n and one of degree k . Then Ψ is homotopic through b-homotopies rel $\partial(D^2 \times [0, 1])$ to a b-homotopy with singular set a 1-manifold.*

Proof. If Ψ_0 and Ψ_1 only have one critical value, then perturb them slightly to branched coverings Ψ'_0 and Ψ'_1 which have two critical values. The b-homotopy from Ψ_i to Ψ'_i has critical locus a 1-manifold. Then, apply Lemma 3.6 to conclude that Ψ'_0 and Ψ'_1 are equivalent by homeomorphisms $f, g : D^2 \rightarrow D^2$ such that

$$g \circ \Psi'_0 = \Psi'_1 \circ f \quad \Rightarrow \quad \Psi'_0 = g^{-1} \circ \Psi'_1 \circ f.$$

Both f and g are isotopic to the identity by isotopies f_t and g_s . Throughout that isotopy, the map $g_s \circ \Psi'_1 \circ f_t$ is a branched covering with one critical point of degree n , one critical point of degree k , and two distinct critical values for all s and t . Thus, there is a b-homotopy from Ψ_0 to Ψ'_1 with singular set a 1-manifold.

Let $H : D^2 \times [0, 1] \rightarrow D^2$ be the stack of the perturbations and this b-homotopy. Then $H|_{D^2 \times \{0\}} = \Psi_0$ and $H|_{D^2 \times \{1\}} = \Psi_1$ and the singular set of H is a 1-manifold. Finally, we apply Lemma 3.4 to conclude that Ψ and H are homotopic through b-homotopies rel $\partial(D^2 \times [0, 1])$. $\square$

3.4. Belyi barycentric branched coverings. We now turn to a specific kind of branched covering which are easily constructed from a triangulation. This will be a key tool in Section 4.

Definition 3.8. A **Belyi map** is a branched covering $\psi : S \rightarrow S^2$ which has exactly 3 critical values.

Remark 2. This is a topological version of a Belyi function—a holomorphic map from a compact Riemann surface to $\mathbb{P}^1$ ramified over $\{0, 1, \infty\}$. In 1979, Belyi showed that a Riemann surface admits a Belyi function if and only if it is a non-singular algebraic curve with algebraic number coefficients. Belyi functions are expressed in a combinatorial way using dessins d'enfants which dates at least to the work of Klein. We use the same checkerboard coloring scheme in this paper.

Construction 3.9. *Suppose that S is a surface with a triangulation τ . Then there is a Belyi map $\psi : S \rightarrow S^2$ naturally associated to τ .*

Construction of 3.9. Let τ' be the barycentric subdivision of τ . Vertices in τ' are either at vertices of τ (0-dim), centers of edges of τ (1-dim), or centers of faces of τ (2-dim). Label each vertex of τ' with $k \in \{0, 1, 2\}$ for the dimension of the corresponding cell in τ . Color in alternating triangles of τ' using the following procedure. For a triangle Δ , if the counterclockwise cyclic order of its vertices is $(0, 1, 2)$, then color Δ black. Otherwise, color Δ white. See Figure 10 for an example.

Next we put a corresponding triangulation on the sphere. Label three marked points on the sphere with p_0, p_1, p_2 . Then add an edge between p_i and p_j for each pair $i \neq j$. These edges divide the sphere into two triangles. Let Δ_B be the triangle with vertices in counterclockwise cyclic order (p_0, p_1, p_2) ; color it black. The other triangle Δ_W is colored white. Finally, define a map $\psi : S \rightarrow S^2$ piecewise on each triangle of τ' . For a triangle $\Delta \in \tau$, map Δ via a diffeomorphism to $\Delta_{B/W}$ depending on the color of Δ . Match up each vertex labeled k with p_k . Then define ψ on the edges so it is a smooth map. Then ψ is a Belyi branched covering. $\triangle$

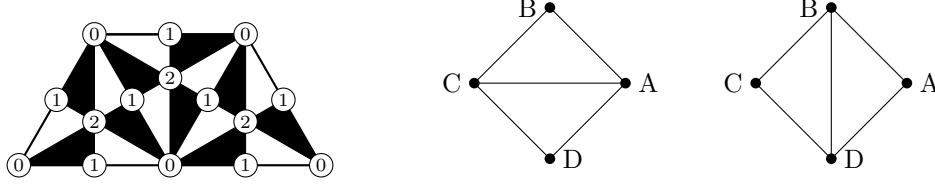


FIGURE 10. (left) A piece of a barycentric subdivision with appropriate coloring. (right) Two triangulations which differ by a 2-2 move.

Definition 3.10. Suppose that τ_0 and τ_1 are two triangulations of a surface S . We say that they *differ by a 2-2 move* if there is a single edge deletion and replacement which turns τ_0 into τ_1 according to the local model in Figure 10 (right). We may also say that their corresponding Belyi maps differ by a 2-2 move.

Proposition 3.11 (2-2 move). *Suppose that τ_0, τ_1 are two triangulations which differ by a 2-2 move and let $\psi_0, \psi_1 : S \rightarrow S^2$ be their corresponding Belyi maps according to Construction 3.9. Then ψ_0 and ψ_1 are b-homotopic.*

Proof. Let $R \subset S$ be the rectangle composed of two triangles from τ_0 and two triangles from τ_1 in which the 2-2 move happens. Label the vertices of R by g_1, g_2, g_3, g_4 so that there is an edge between g_1 and g_3 in τ_0 and an edge between g_2 and g_4 in τ_1 . Note that each g_j is a critical point of the maps ψ_0 and ψ_1 . The other critical points are labeled by b_j and r_j in Figure 11.

Let $p_0 := \psi_i(g_j)$ be the critical value for all g_j and let $\alpha \subset S^2$ be a small circle centered at p_0 . Then $\psi_i^{-1}(\alpha)$ has a connected component encircling g_j for each $j = 1, \dots, 4$. Let $\alpha_{i,j}$ be this component and let $D_{i,j}$ be the disk it bounds containing g_j . Note that $\psi_i|_{\alpha_{i,j}}$ is a k -fold covering map onto α ; $k \geq 2$ is the valence of g_j in the triangulation τ_i .

Suspend ψ_0 to a b-homotopy $\hat{\psi}_0$ and insert the splitting homotopy σ in $D_{0,1} \times [0, 1]$ and $D_{0,3} \times [0, 1]$ according to Construction 3.2, splitting critical points g_1 and g_3 . Let G be the resulting b-homotopy. Do this so that $G_0|_R$ and $G_1|_R$ are the corresponding labeled maps defined by Figure 11. Repeat the analogous construction to split critical points g_2 and g_4 for ψ_1 and obtain a b-homotopy H so that $H_1|_R$ and $H_0|_R$ are the maps labeled in Figure 11.

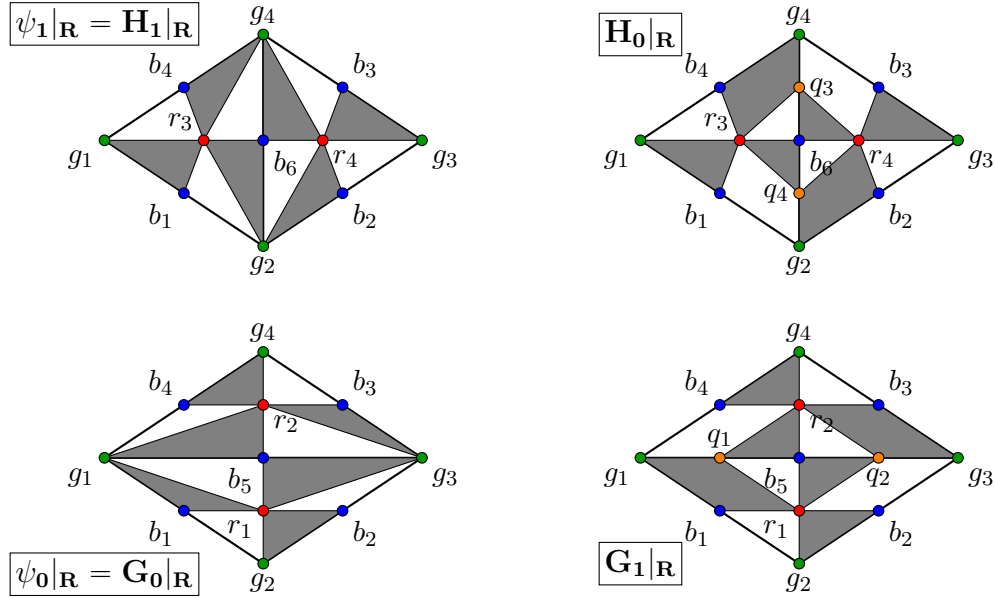


FIGURE 11. (left) A picture of two Belyi branched maps $\psi_0|_R$ and $\psi_1|_R$ which differ a 2-2 move in a rectangle. (right) The two maps on the left after performing a splitting b-homotopy. These are $G_1|_R$ and $H_0|_R$.

Now we build a b-homotopy between G_1 and H_0 and we stack them all. We still let p_0 be the critical value of g_j under G_1 and H_0 . Also let p_1 be the critical value of b_j under these maps. Let $\gamma \subset S^2$ be a simple closed curve in S^2 encircling p_0 and p_1 and no other critical values. Then $G_1^{-1}(\gamma)$ and $H_0^{-1}(\gamma)$ each have a component parallel to ∂R . Up to a small perturbation, we can assume that these are the same simple closed curve and call it $\tilde{\gamma}$. Let $D \subset S$ be the disk bounded by $\tilde{\gamma}$ which is contained in R .

By examining the triangulations in Figure 11, we see that $G_1|_{\tilde{\gamma}}$ and $H_0|_{\tilde{\gamma}}$ are both 4-fold covering maps and are therefore equal up to isotopy. By Lemma 3.2 in [3] and the additional discussion in Section 3.3 of this paper, we conclude that there is a b-homotopy between $G_1|_D$ and $H_0|_D$. Insert this into the suspension b-homotopy of G_1 to obtain a b-homotopy $K : S \times [0, 1] \rightarrow S^2$ between G_1 and H_0 . Finally, we stack $G \cdot K \cdot H$ and this is a b-homotopy from ψ_0 to ψ_1 . $\square$

In the proof of Proposition 3.11, we appealed to the existence of a b-homotopy between G_1 and H_0 , but we did not give an explicit construction. In order to have control over the singular set in Section 4, we give two explicit constructions of this b-homotopy.

Consider the two b-homotopies $\rho, \lambda : \bar{R} \times [0, 1] \rightarrow S^2$ shown in Figure 12. They are constructed by a sequence of isotopies rel ∂ , splitting critical points, combining critical points, and retriangulation. These two maps differ by how they connect the critical points at the centers of faces on $R \times \{0\}$ and $R \times \{1\}$. Also, notice that $\rho_0 = \lambda_0 = G_1|_R$ and $\rho_1 = \lambda_1 = H_0|_R$ so they can be inserted in place of K in the proof of Proposition 3.11.

3.5. Restricting a branched submersion to a surface. We now cover a technical point that will be relevant for concluding that a branched submersion is a leafwise branched covering

for a given foliation. Consider a constant b-homotopy on the unit disk $\mathbb{D} \subset \mathbb{C}$.

$$\Psi_c : \mathbb{D} \times I \rightarrow \mathbb{D}, \quad \Psi_c(z, t) = z^2$$

The singular set is $X(\Psi_c) = \{(0, t) \mid t \in I\}$ and at each point $q \in X(\Psi_c)$, we have $D_q \Psi_c = 0$, the zero map. For any surface $\Sigma \hookrightarrow \mathbb{D} \times I$ transversely intersecting fibers of Ψ_c and $X(\Psi_c)$, the restriction $\Psi_c|_\Sigma$ is a branched immersion. This follows because $T_q \Sigma \subset \ker D_q \Psi_c = T_q M$ for each $q \in \Sigma \cap X(\Psi_c)$. To see why this condition is necessary, we consider the following b-homotopy.

$$\Psi_n : \mathbb{D} \times I \rightarrow \mathbb{D}, \quad \Psi_n(x, t) = z^2 + t$$

The singular set is still $X(\Psi_n) = \{(0, t) \mid t \in I\}$. For this map, the derivative $D_{(0,t)} \Psi_n : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ is given by the following matrix.

$$D_{(0,t)} \Psi_n = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

By explicit check, one can see that if a surface $\Sigma \hookrightarrow \mathbb{D} \times I$ does not satisfy $T_q \Sigma \subset \ker D_q \Psi_n$ (it is not “horizontal” in this coordinate system), then $\Psi_n|_\Sigma$ will not be a branched immersion. However, this can be fixed with a slight isotopy of either Σ or Ψ_n .

From these examples, we conclude that if $\Psi : M \rightarrow S$ is a branched submersion and $\mathcal{F}$ is a foliation transverse to the fibers outside of $X(\Psi)$ and transverse to $X(\Psi)$, we may need to insert a slight isotopy of either $\mathcal{F}$ or Ψ to guarantee that $T_q \mathcal{F} \subset D_q \Psi$ for $q \in X(\Psi)$. This will then imply that Ψ is a leafwise branched covering for $\mathcal{F}$. This will be relevant for Corollary 4.3.

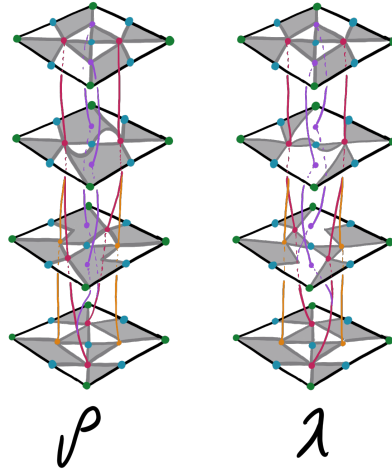


FIGURE 12. Two options for an explicit homotopy between $\overline{\psi_1}$ and $\overline{\psi_2}$ which differ by a 2-2 move. ρ moves the pink critical points clockwise from bottom to top and λ moves them counterclockwise.

4. CONSTRUCTING THE MAP

We are now ready to construct the map referenced in Theorem A. In fact, we will start by constructing a branched submersion $\psi : M \rightarrow S^2$ depending only the pseudo-Anosov flow

and not the foliation. Then we will see that this map is a leafwise branched covering for every taut foliation transverse to the flow.

We defined branched submersions in Section 3.1. Such a map ψ has a singular set $X(\psi)$. Away from $X(\psi)$, the map ψ is a submersion; around the singular set, ψ is a b-homotopy of branched immersions

Let M be a compact 3-manifold, possibly with toral boundary, with a pseudo-Anosov flow φ . Let κ be a finite set of closed orbits of φ which kill its perfect fits and let κ_S be the union of this set with the singular orbits, as in Section 2.1. Then define $\mathring{M} = M \setminus \kappa_S$ and let τ be the veering triangulation of $\mathring{M}$ given by the construction in Section 2. The 2-skeleton of τ is a branched surface denoted $\tau^{(2)}$.

Theorem 4.1. *For M , φ , and τ as above, there is a branched submersion $\psi : M \rightarrow S^2$ such that $\psi|_{\tau^{(2)}}$ is a branched immersion on every sector and $X(\psi)$ is a union of closed orbits of φ containing all the singular orbits.*

Proof. We start by constructing the map ψ and then we will see that it satisfies the stated properties. Note that the 2-skeleton $\tau^{(2)}$ is a branched surface with a natural division into triangles by the 2-faces of τ . The ideal vertices lie at drilled orbits of M . We then define $\psi^{(2)} : \tau^{(2)} \rightarrow S^2 \setminus \{p_0\}$ to be the Belyi map associated to this triangulation. The ideal vertices of triangles are sent to a puncture at $p_0 \in S^2$. We will then extend $\psi^{(2)}$ to be defined on all of M . Recall that in Section 2.4, we picked out closed cycles of the dual graph Γ and the flow graph Φ which we called $c_1, \dots, c_n$ and $d_1, \dots, d_k$, respectively. For Γ , this was done by making a choice of resolution at each vertex.

Claim 4.2. *There is a branched submersion $\psi' : M \rightarrow S^2$ which extends $\psi^{(2)}$ such that ψ' is homotopic to a branched submersion ψ'' satisfying*

- (1) $\psi''|_{\tau^{(2)}}$ is a branched immersion on every sector
- (2) and the singular set $X(\psi'')$ is 1-manifold with each component homotopic to either some c_i or some d_i .

Step 1: Constructing ψ'

To construct this extension, within each tetrahedron Δ of τ we will insert a branched submersion $\text{int}(\Delta) \rightarrow S^2$ whose singular set is combinatorially coded in the same way as the segments of c_i and d_i inside of Δ . That is, the segments of the singular set will enter and exit Δ through the same faces or edges of Δ as some segment of c_i or d_i .

Let R be a closed rectangle with vertices labeled $g_1, \dots, g_4$ as in Figure 11. We identify $\text{int}(\Delta)$ with $\text{int}(R \times [0, 1])$ such that the two bottom faces of Δ share vertices g_1, g_3 and the two top faces of Δ share vertices g_2, g_4 . In the proof of Proposition 3.11, we defined two Belyi maps $\psi_i|_R : R \rightarrow S^2$ differing by a 2-2 move. Notice that

$$\psi^{(2)}|_{R \times \{0\}} = \psi_0|_R \quad \text{and} \quad \psi^{(2)}|_{R \times \{1\}} = \psi_1|_R.$$

From Proposition 3.11, we know these two maps are b-homotopic relative to ∂R . Here, we will construct a specific b-homotopy $F : R \times [0, 1] \rightarrow S^2$ with $F_i = \psi_i|_R$ ($i = 0, 1$) depending on the resolution within tetrahedron Δ .

Let c_1, c_2, d be the components of $(\bigcup c_i \cup \bigcup d_i) \cap \Delta$ such that $(c_1 \cup c_2) \subset \bigcup c_i$ and $d \subset \bigcup d_i$. The singular set of F will be a directed graph (directed by the orientation of the flow) with endpoints at the critical points of $\psi_0|_R$ and $\psi_1|_R$. We will define F so that all directed paths in the singular set are homotopic rel endpoints to one of c_1, c_2 , or d . We start with b-homotopies $G|_{R \times [0, 1]}$ and $H|_{R \times [0, 1]}$ defined in the proof of Proposition 3.11; the maps $G_1|_R$

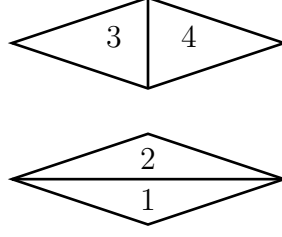


FIGURE 13. Labeling of the 2-faces of Δ . Coorientation is from the bottom to top of the picture.

and $H_0|_R$ are shown in Figure 11. Recall that G splits critical points g_1 and g_3 of ψ_0 and H splits critical points g_2 and g_4 of ψ_1 . Up to isotopy, we can assume that G_t and H_t are constant on ∂R . The next step is to fill in with a b-homotopy between $G_1|_R$ and $H_0|_R$ which has a singular set matching the resolution of Γ in Δ .

The resolution of Γ in Δ determines a pairing of the bottom faces of Δ with the top faces. Label the faces according to Figure 13. The way that these faces are paired depends on whether the resolution is branching/anti-branching and the veer of the top edge. We summarize this pairing in the table below. The segments c_1 and c_2 inside Δ will intersect the faces of Δ in these face pairings.

Resolution of Δ	veer of top edge	
	Red	bLue
branching (B)	$1 \leftrightarrow 3, 2 \leftrightarrow 4$	$1 \leftrightarrow 4, 2 \leftrightarrow 3$
antibranching (AB)	$1 \leftrightarrow 4, 2 \leftrightarrow 3$	$1 \leftrightarrow 3, 2 \leftrightarrow 4$

Consider now the b-homotopies ρ and λ defined at the end of Section 3.4 and shown in Figure 12. The singular set of these maps pair up the critical points of $G_1|_R$ and $H_0|_R$, labeled in Figure 11 by $\{r_1, r_2, q_1, q_2, b_5\}$ and $\{r_3, r_4, q_3, q_4, b_6\}$, respectively.

- **b_i endpoints:** In both ρ and λ , there is a segment of the singular set connecting b_5 to b_6 . This is homotopic rel endpoints to d by construction.
- **r_i endpoints:** Note that ρ pairs $r_1 \leftrightarrow r_3$ and $r_2 \leftrightarrow r_4$ while λ pairs $r_1 \leftrightarrow r_4$ and $r_2 \leftrightarrow r_3$ with segments of the singular set. Depending on the face pairing determined by Table ??, we will choose either ρ or λ .
- **q_i endpoints:** Both ρ and λ have the same segments of the singular set with the q_i 's as endpoints. They pair

$$q_1 \leftrightarrow r_3, \quad q_2 \leftrightarrow r_4, \quad r_2 \leftrightarrow q_3, \quad r_1 \leftrightarrow q_4.$$

It is not clear how these segments are encoded by paths in Γ since they have endpoints on edges of $\tau^{(2)}$. We then need to perturb $G_1|_R$ and $H_0|_R$ so that $q_1, \dots, q_4$ all lie in the interior of faces of $\tau^{(2)}$.

To do this, consider a tetrahedron Δ' glued along a top face of Δ . It will be attached in one of two ways, depending on the veer of the top edge of Δ as shown in Figure 14. Within these two tetrahedra, we look at the segment in the singular set passing through q_4 (yellow curve in Figure 14). Notice that this segment is homotopic to a curve which intersects no edge of τ and is then encoded by a path in Γ (orange curve in Figure 14). From this observation, we perturb the maps $G_1|_R$ and $H_0|_R$ depending on the veer of the middle edge to assign $q_1, \dots, q_4$ to faces of $\tau^{(2)}$. If the middle edge is red, we perform a counterclockwise ‘twist’; if the middle edge is blue,

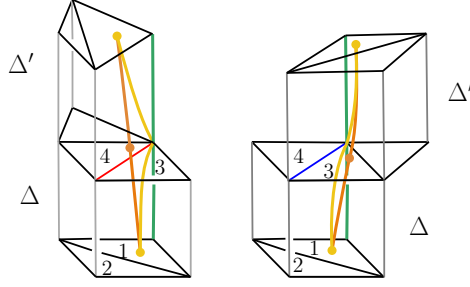


FIGURE 14. The segment of the singular set passing through q_4 (yellow segment) is homotopic to one which intersects no edge of τ (orange segment). This homotopy differs based on the veer of the middle edge.

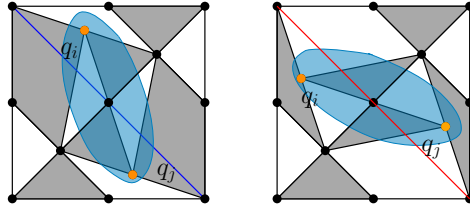


FIGURE 15. The maps $G_1|_R$ and $H_1|_R$ will be perturbed depending on the veer of the middle edge. A do-si-do⁺ b-homotopy may then need to be inserted. The support of this homotopy is the blue disk.

we perform a clockwise ‘twist’. These perturbations are shown Figure 15. Abusing notation, we will still call these maps $G_1|_R$ and $H_0|_R$. The assignment to faces is summarized in the following table.

	Red	bLue
top edge	$q_3 \mapsto 3$	$q_3 \mapsto 4$
	$q_4 \mapsto 4$	$q_4 \mapsto 3$
bottom edge	$q_1 \mapsto 1$	$q_1 \mapsto 2$
	$q_2 \mapsto 2$	$q_2 \mapsto 1$

To ensure that the segments of the singular set with q_i as endpoints match c_1 or c_2 , we may need to insert a do-si-do⁺ b-homotopy δ^+ once or twice. Inserting before ρ/λ will switch the places of q_1 and q_2 ; inserting after ρ/λ will switch the places of q_3 and q_4 . In Figure 15, the support of δ^+ is encircled by a turquoise curve.

We summarize the choice of b-homotopy from $G_1|_R$ to $H_0|_R$, for four kinds of tetrahedra depending on the veer of the top edge (red/blue) and the type (toggle/fan). Recall that in the resolution chosen by Proposition 2.11, fan tetrahedra will always be resolved anti-branching and toggle tetrahedra may be resolved either branching or anti-branching. The b-homotopy F will then be $G|_{R \times [0,1]} \cdot \nu \cdot H|_{R \times [0,1]}$ where ν is chosen appropriately from the table below.

	toggle, red	toggle, blue	fan, red	fan, blue
branching (B)	$\delta^+ \cdot \rho \cdot \delta^+$	$\delta^+ \cdot \lambda \cdot \delta^+$		
anti-branching (AB)	λ	ρ	$\delta^+ \cdot \lambda$	$\delta^+ \cdot \rho$

We now define $\psi' : M \rightarrow S^2$ on all of M . On $\tau^{(2)}$, we set $\psi' = \psi^{(2)}$. Then, for a given tetrahedron Δ , choose $F : \text{int}(\Delta) \rightarrow S^2$ according to the instructions above. Since F is a

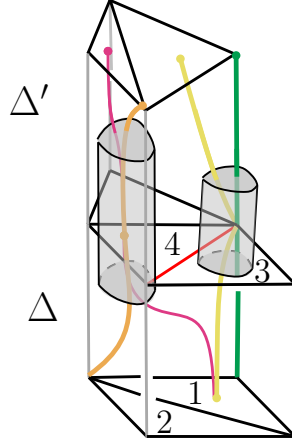


FIGURE 16. Two tetrahedra Δ and Δ' which share a face and the relevant part of $X(\psi')$ which will be split apart. Within the gray cylindrical neighborhoods, we apply a homotopy through b-homotopies.

b-homotopy rel ∂R , it extends continuously to $\partial\Delta$. On the locus of drilled orbits κ_S , ψ' is the constant map to p_0 .

Step 2: Homotope ψ' to ψ''

Next we perform a homotopy of ψ' through branched submersions to $\psi'' : M \rightarrow S^2$ with singular set a 1-manifold such that each component of $X(\psi'')$ is homotopic to some c_i or d_i . We will perform this homotopy locally. Consider two tetrahedra Δ and Δ' glued along a face. The singular set $X(\psi')$ has two non 1-manifold pieces which combine and split within $\Delta \cup \Delta'$. Figure 16 shows this in the case where the top edge of Δ is red. Both of these pieces have a neighborhood homeomorphic to $D^2 \times [0, 1]$ on which ψ' is modeled by a b-homotopy $\mu : D^2 \times [0, 1] \rightarrow D^2$ of branched coverings of the disk where μ_0 and μ_1 are branched coverings with two critical points with the same degree set. These neighborhoods are also shown in Figure 16. We then apply Corollary 3.7 to perform a homotopy of μ rel $\partial(D^2 \times [0, 1])$ through b-homotopies which splits the singular set into a 1-manifold. We will repeat this procedure finitely many times for each pair of tetrahedra in τ glued along a face. Let ψ'' be the resulting map after these homotopies.

For a small enough choice of neighborhoods, the components of $X(\psi'')$ will intersect the same sequence of faces as the corresponding segments of $X(\psi')$. Thus, each component will be homotopic to some c_i or d_i by the construction of ψ' . Also, this local homotopy ensures that $\tau^{(2)}$ will still be transverse to fibers of ψ'' and $X(\psi'')$, so $\psi''|_{\tau^{(2)}}$ is a branched immersion on every sector.

Step 3: Homotope ψ'' to ψ

We now perform a homotopy of ψ'' through branched submersions to ψ so that $X(\psi)$ is exactly a union of closed orbits of φ . We address each component α of $X(\psi'')$ sequentially.

Take an annulus $S^1 \times [0, 1]$ bounding α and a closed orbit γ . Let $U = D^2 \times S^1$ be a tubular neighborhood of α . Split U into finitely many copies of $D^2 \times [0, 1]$. Within each of

these, apply Lemma 3.3 to homotope α to $S^1 \times \{\varepsilon\}$ rel $\partial D^2 \times [0, 1]$. We can do this in a consistent way on the boundary disks $D^2 \times \{0\}$ and $D^2 \times \{1\}$. Then repeat the process with a new tubular neighborhood and build a homotopy through branched submersions which homotopes α to γ .

If some other singular set component α' intersects the annulus, the degree of the map within some $D^2 \times [0, 1]$ may change. Up to an isotopy, we may assume α' intersects the foliation on $D^2 \times S^1$ transversely. We can then continue the homotopy through this intersection. Note that for a moment, two singular set components may necessarily intersect.

Since there are finitely many components of $X(\psi'')$, we perform each of these homotopies in sequence to obtain ψ . Note that multiple components of $X(\psi'')$ may be homotoped into the same closed orbit. This will only increase the degree of branching around that curve.

By Landry-Minsky-Taylor [14], we can execute these homotopies while keeping the singular set transverse to $\tau^{(2)}$ at all times. This guarantees that $\psi|_{\tau^{(2)}}$ is a branched immersion on every sector. In the end, ψ has $X(\psi)$ exactly a union of closed orbits containing all the singular orbits of φ . $\square$

Corollary 4.3. *If $\mathcal{F}$ is a foliation transverse to a pseudo-Anosov flow φ and $\psi : M \rightarrow S^2$ is the branched submersion constructed in the proof of Theorem 4.1, then, up to isotopy, ψ is a leafwise branched covering for $\mathcal{F}$.*

Proof. Let τ be the veering triangulation from which ψ was constructed. In “Veering triangulations and transverse foliations” [20], Zung shows that $\mathcal{F}$ splits open into a lamination $\mathcal{L}$ fully carried by $\tau^{(2)}$. By construction, $\tau^{(2)}$ is transverse to the fibers $\psi^{-1}(p)$, except possibly at $q \in X(\psi)$, and transverse to the singular set $X(\psi)$. Also, by construction, for each sector $\sigma \in \tau^{(2)}$ and $q \in X(\psi) \cap \sigma$, we have $T_q\sigma \subset \ker D_q\psi$. Then, up to an isotopy, we can assume $\mathcal{L}$ satisfies these conditions as well. Within each tetrahedron, blow down $\mathcal{L}$ to $\mathcal{F}$ while maintaining transversality and the $\ker D_q\psi$ condition. This implies that, up to isotopy, $\mathcal{F}$ is transverse to both these structures and $\psi|_\lambda$ is a branched covering for every $\lambda \in \mathcal{F}$. $\square$

5. TAUT CONE FIELDS

The goal of this section is to use language from Lorentzian causality theory to reinterpret branched submersions, their related flows, and transverse surfaces with the aim to define taut cone fields. We see these as an analogue for taut foliations and an additional low-dimensional structure for a three-manifold. For more information on Lorentzian manifolds and the following definitions, see Minguzzi’s review article on Lorentzian causality theory [16]. The following definitions and propositions are stated in this source.

Let (M, g) be a closed connected 3-manifold with a Lorentzian metric (i.e. g is a symmetric bilinear form with signature $(-, +, +)$). A vector $v \in T_p M$ is

- **timelike** if $g_p(v, v) < 0$,
- **lightlike** if $g_p(v, v) = 0$ and $v \neq 0$,
- **spacelike** if $g_p(v, v) > 0$ or $v = 0$.

We can extend this definition to submanifolds of M as well. A submanifold $N \subset M$ is timelike/lightlike/spacelike if $T_p N$ is timelike/lightlike/spacelike for every $p \in N$. Since M is 3-dimensional, it will have timelike curves and spacelike surfaces. We call a flow on M timelike if the flow lines are all timelike.

A **causal** vector is either lightlike or timelike. The set of all causal vectors at a point $p \in M$ form a cone C_p . The assignment of this cone to each $p \in M$ defines a **cone field** $\mathcal{C}$ on

M . This is then a low-dimensional *topological* structure on M which comes from a choice of Lorentzian metric.

Assume that there is a global **time-orientation** on M . This is a consistent choice of component of C_p to be the **future direction**. This definition is analogous to a co-orientation on a foliated manifold. Define the **recurrence set** (called the **chronology violating set** in [16]) to be

$$\mathcal{R} = \{p \in M \mid \text{there is a closed timelike curve through } p\}.$$

Definition 5.1. A Lorentzian manifold (M, g) is **totally vicious** if $M = \mathcal{R}$.

In Proposition 4.20 of [16], Minguzzi proves that (M, g) is totally vicious if and only if there is a future-directed timelike curve from any $p \in M$ to any $q \in M$.

Proposition 5.2. *If (M, g) is a Lorentzian manifold with a timelike transitive flow φ , then (M, g) is totally vicious.*

Proof. Since φ is transitive, it has a dense set of closed orbits which are all timelike. For $p \in M$, find a timelike closed orbit passing arbitrarily close to p . Take a parallel copy which passes through p and this will also be timelike by continuity of g and openness of the timelike condition. $\square$

Also, we observe that if a foliation is spacelike for a totally vicious Lorentzian manifold, then the closed timelike curves through every point certify tautness for the foliation.

Proposition 5.3. *If $\mathcal{F}$ is a spacelike foliation for a totally vicious Lorentzian manifold (M, g) , then $\mathcal{F}$ is taut.*

Both of the results above are in the same spirit as the fact that a foliation transverse to a transitive flow is taut. We begin to see that transitivity of a flow, tautness of a foliation, and totally vicious Lorentzian metrics are all related concepts.

A **Lorentzian submersion**, as defined by Flaherty in [11], is a submersion $\Psi : (M, g) \rightarrow (S, h)$ where (M, g) is a Lorentzian 3-manifold, (S, h) is a Riemannian 2-manifold, and the fibers $\Psi^{-1}(b)$ for $b \in S$ are timelike. We extend this definition to allow for branching along timelike curves.

Definition 5.4. A **branched Lorentzian submersion** is a branched submersion $\Psi : (M, g) \rightarrow (S, h)$ such that the fibers $\Psi^{-1}(b)$, $b \in S$ and the singular set $X(\Psi)$ are timelike curves.

Immediately, we see that if (M, g) admits a branched Lorentzian submersion Ψ , then any spacelike surface Σ will be transverse to both the fibers and the singular set of Ψ . Thus, $\Psi|_{\Sigma}$ is a branched immersion. This means that Ψ is a leafwise branched covering for every spacelike foliation. This observation is an indicator of the following result

Proposition 5.5. *A Lorentzian 3-manifold (M, g) admits a branched Lorentzian submersion to S^2 if and only if (M, g) is totally vicious.*

Proof. For the forward direction, let $\Psi : M \rightarrow S^2$ be a branched Lorentzian submersion and consider a fiber $\Psi^{-1}(b)$ for $b \in S^2$. If $b \notin \Psi(X(\Psi))$, then $\Psi^{-1}(b)$ is a closed 1-manifold whose intersection with every spacelike surface is discrete. We conclude that $\Psi^{-1}(b)$ is a union of timelike circles. If $b \in \Psi(X(\Psi))$, then $\Psi^{-1}(b)$ may be a wedge of circles, but it will still contain a closed timelike curve through every point. Thus, (M, g) is totally vicious.

For the backward direction, assume that (M, g) is totally vicious and fix a Riemannian metric on M . Note that we assume M is compact or, at worst, precompact. Choose enough

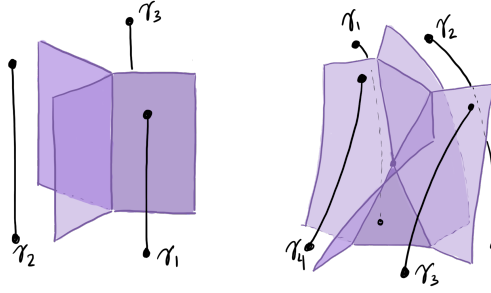


FIGURE 17. Three solid tori will meet around an edge (left), and four solid tori will meet around a vertex (right).

timelike closed curves on M so that open solid torus neighborhoods form an open cover of M . Then pick a finite subcover and let $\gamma_1, \dots, \gamma_n$ be the corresponding closed curves. Then let $T_i \subset M$ be the set of all points closest to γ_i . Note that T_i deformation retracts to γ_i so it is an open solid torus. Then $\overline{T_i}$ glue together along their boundaries to form M . Put $\gamma_1, \dots, \gamma_n$ into general position so that each of $\partial \overline{T_i}$ has finitely many vertices and edges. Do this so edges of $\partial \overline{T_i}$ are shared by three of $\{T_i\}$ and vertices of $\partial \overline{T_i}$ are shared by four of $\{T_i\}$. See Figure 17 for a local picture of these configurations.

Each of $T_1, \dots, T_n$ is a solid torus with a polyhedral boundary. Each T_i has a core curve γ_i which is a closed timelike curve with a future orientation. The faces of $\partial \overline{T_i}$ are either annuli with the S^1 direction parallel to γ_i or polygons with a unique “past-most” vertex v_p and a unique “future-most” vertex v_f .

Each 2-face σ has two solid tori $\overline{T_i}$ and $\overline{T_j}$ glued along σ . We insert a 2-cell $\Delta_\sigma : [0, 1]^2 \rightarrow M$ transversely intersecting σ so that $\{0\} \times [0, 1]$ lies in γ_i and $\{1\} \times [0, 1]$ lies in γ_j . If σ is an annulus, then Δ_σ is a transverse annulus to σ bounding γ_i and γ_j . If σ is a polygon, then Δ_σ is an embedding of $[0, 1]^2$. The local picture around a vertex is shown in Figure 18a.

Next, we insert a 2-cell “above” and “below” each vertex v_p or v_f of a polygon face. These are shown in Figure 18b. In the result, we have a cell complex for M so that each 3-cell is a triangular prism or a rectangular prism triangulated as in Figure 18c.

We then define a branched submersion $\Psi : M \rightarrow S^2$ piecewise on each 3-cell so that the boundaries match up. On each triangular prism, insert the suspension of the barycentric Belyi map as described in Sections 3.4 and 3.2. Within the other 3-cells, we have defined Ψ on their top and bottom faces to be maps ψ_0 and ψ_1 as described in Section 3.4. Then we can insert a b-homotopy between these maps as described in that section. This builds a piecewise branched submersion on M . Up to a homotopy, we can guarantee that the fibers and singular set are timelike curves.

□

From Proposition 5.5, we are motivated to make the following definition. For a general cone field $\mathcal{C}$ on M , possibly not associated with a Lorentzian metric, we define a timelike curve γ to have $T_p \gamma \in \text{int}(C_p)$ for each $p \in \gamma$.

Definition 5.6. A cone field $\mathcal{C}$ on M is **taut** if there is a closed timelike curve through every point. If $\mathcal{C}$ comes from a Lorentzian metric g on M , then $\mathcal{C}$ is taut if (M, g) is totally vicious.

For a general cone field, we can also define a branched Lorentzian submersion in the same way as Definition 5.4.

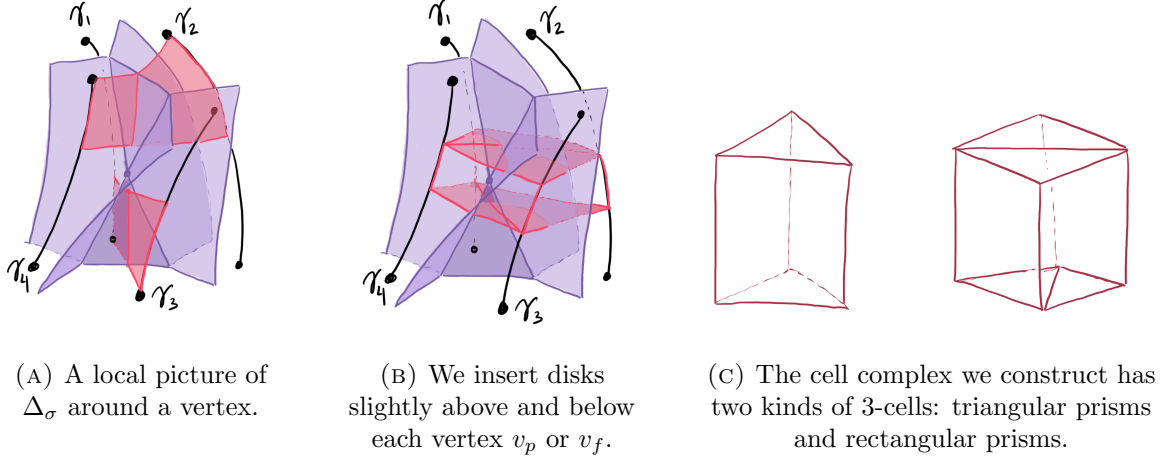


FIGURE 18. We construct a 2-complex on M with 2-cells transversely intersecting $\partial\overline{T}_i$.

Remark 3. There is an immediate example of a non-taut cone field. Take a foliation $\mathcal{F}$ which is not taut (for example, a Reeb foliation) and then choose a cone field $\mathcal{C}$ for which $\mathcal{F}$ is spacelike. There can be no leafwise branched covering to S^2 for $\mathcal{F}$ since it is not taut. But any branched Lorentzian submersion for $\mathcal{C}$ would be a leafwise branched covering for $\mathcal{F}$. So $\mathcal{C}$ cannot be taut.

Now we will use this framework to rephrase Theorem 4.1.

Theorem 5.7. *If φ is a pseudo-Anosov flow on compact 3-manifold M , then there is a taut cone field $\mathcal{C}$ on M for which φ is timelike and there is a branched Lorentzian submersion $\Psi : M \rightarrow S^2$ with $X(\Psi)$ a union of closed orbits of φ , containing all the singular orbits. Any foliation transverse to φ can be isotoped to be spacelike with respect to this cone field.*

In the proof of Theorem 4.1, we constructed the desired branched submersion. Then we can choose the cone field so that φ , $X(\Psi)$, and the fibers of Ψ are all timelike, and the branched surface $\tau^{(2)}$ is spacelike.

The point of introducing this structure is to bundle together the three one dimensional structures—flow lines, fibers of Ψ , and $X(\Psi)$ —to which a surface can be transverse. By insisting that these are all timelike for some cone field, we ensure that any spacelike surface Σ is transverse to them all and that $\Psi|_\Sigma$ is a branched immersion.

6. EXAMPLE: THE FIGURE EIGHT KNOT COMPLEMENT

We will now apply the construction in the proof of Theorem 4.1 to produce a branched submersion on the figure eight knot complement catered to the suspension flow of the fibration.

Let K be the figure eight knot (4_1 in Rolfsen notation) and let $M = S^3 \setminus K$ be its complement. It is well-known that M fibers over S^1 with a punctured torus fiber. Parameterize the torus T by $(\mathbb{R}^2/\mathbb{Z}^2) \setminus \{(0,0)\}$ and then the monodromy f of the fibration is given by the following matrix, acting by linear transformation on $\mathbb{R}^2/\mathbb{Z}^2$.

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

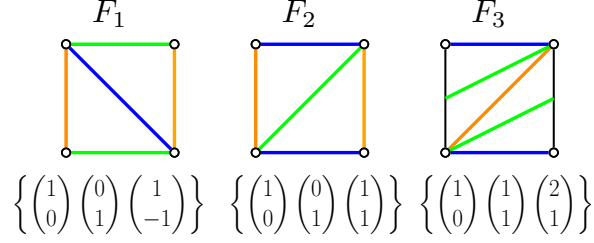


FIGURE 19. Three triangulations of a punctured torus and the corresponding vectors for the slopes of the edges.

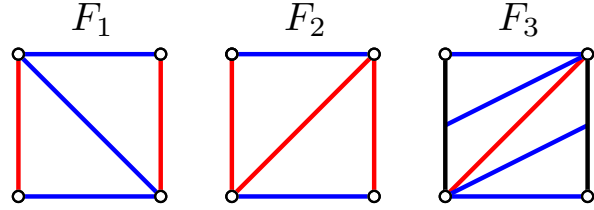


FIGURE 20. Color the edges of the triangulations F_1, F_2, F_3 according to two rules and obtain two equivalence classes of edges.

The foliation $\mathcal{F}$ will be this fibration and the pseudo-Anosov flow will be the corresponding suspension flow. We can reconstruct the veering triangulation of M consisting of two tetrahedra in the following way.

Consider the three triangulations of T shown in Figure 19. Observe that applying the matrix $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ to F_1 returns F_2 and applying $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ to F_2 returns F_3 . So the triangulations F_1 and F_3 differ by the monodromy map f .

Now consider $T \times [0, 1]$ and note that M is homeomorphic to $(T \times [0, 1]) / (x, 1) \sim (f(x), 0)$. Triangulate $T \times \{1\}$ by F_1 and $T \times \{0\}$ by F_3 . Under the identification in M , the orange, green, and blue edges in F_1 on $T \times \{1\}$ are identified with their corresponding color in F_3 on $T \times \{0\}$. We will now (re)color the edges of F_1, F_2, F_3 to identify them. We will follow the following two rules.

- (1) Between F_i and F_{i+1} ($i = 1, 2$), there are two edges the same and one different. Identify the ones which are the same.
- (2) Identify e in F_1 with $f(e)$ in F_3 .

This results in two equivalence classes of edges, shown with red and blue in Figure 20.

Now consider two ideal tetrahedra T_1 and T_2 pictured in Figure 23. We identify the two bottom faces of T_1 with $T \times \{0\}$ along the triangulation F_3 . Then we glue the bottom faces of T_2 to the top faces of T_1 according to the identifications indicated in Figure 23. Then the two top faces of T_2 are identified with $T \times \{1\}$ along triangulation F_1 . This decomposes M as the gluing to two ideal tetrahedra. This is the veering triangulation of M .

We apply the procedure outlined in Propositions 2.10 and 2.11 to pick out closed cycles c_i and d_i . Starting with the flow graph, let d be the directed cycle composed of two $bot \rightarrow top$ edges, one in each of T_1 and T_2 . This cycle is shown in green in Figure 23. We know that d is homotopic to an orbit of the suspension flow; consider the following period 3 orbit in T under the monodromy f .

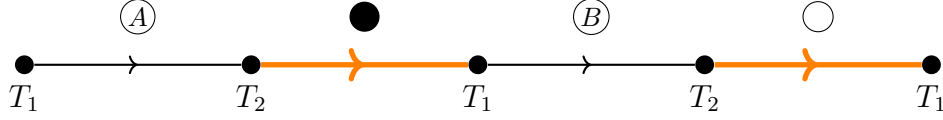


FIGURE 21. The resolution of Γ which is branching for T_1 and antibranching for T_2 produces one directed cycle c given by the above path.

$$\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \mapsto \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} \mapsto \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

All of the points in this orbit lie at the centers of edges in the triangulation F_1 . The suspension of this point is a closed orbit of the suspension flow which is combinatorially coded by d . Call this orbit δ .

We now turn to the dual graph shown in Figure 24. The vertices are labeled by the two tetrahedra and the edges are labeled by the four 2-faces. Both of the tetrahedra are toggle, and so we choose a resolution which is branching for T_1 and anti-branching for T_2 . This choice results in one directed cycle c of Γ . Starting in tetrahedron T_1 , c intersects the 2-faces of τ in the order $A, \bullet, B, \circ$.

Now we would like to find a periodic point under f whose orbit intersects the 2-faces in this same order. When c passes from T_1 to T_2 , it is via suspension in $T \times [0, 1]$, but when it passes from T_2 to T_1 , it is via the monodromy action. These segments of c where f applies are colored orange and bolded in Figure 21. We then see that c wraps twice in the circle fibration direction of M . To find a genuine orbit encoded by c , we look for a period 2 point under the action of f on T . One such orbit is below.

$$\begin{pmatrix} \frac{2}{5} \\ \frac{4}{5} \end{pmatrix} \mapsto \begin{pmatrix} \frac{3}{5} \\ \frac{1}{5} \end{pmatrix} \mapsto \begin{pmatrix} \frac{2}{5} \\ \frac{4}{5} \end{pmatrix}$$

Then we need to confirm that the suspension of $(\frac{2}{5}, \frac{4}{5})$ does indeed intersect the 2-faces of τ in the same order as c . We can work this out explicitly like in Figure 22. Then we can conclude this closed orbit is homotopic to c ; call it γ .

We can then apply the construction in Section 4 to build a branched submersion $\psi : M \rightarrow S^2$. We find that the singular set should have degree 2 branching over δ , degree 4 branching over γ , and degree 4 branching over the boundary curve. We can intersect this singular set with each fiber λ . From this, we conclude that $\psi|_\lambda$ is a degree 8 branched cover to $S^2 \setminus \{p_0\}$ with critical points r_1, r_2, b_1, b_2, b_3 . The local degrees are $\deg(\psi|_\lambda; r_i) = 4$, $\deg(\psi|_\lambda; b_i) = 2$, and $\psi|_\lambda$ has degree 4 branching around the puncture. There are two critical values in S^2 given by p_1, p_2 such that $\psi(b_i) = p_1$ and $\psi(r_i) = p_2$.

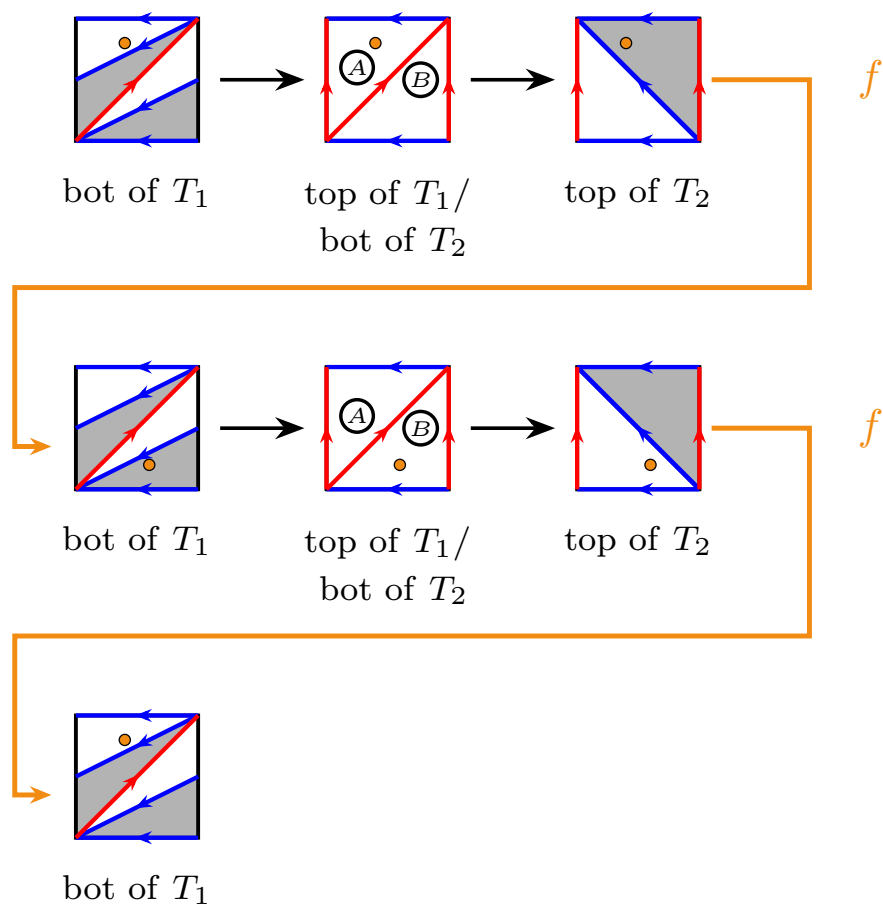


FIGURE 22. The orbit of $(2/5, 4/5)$ under f intersects the 2-faces of τ in the order $\circ, A, \bullet, B, \circ$.

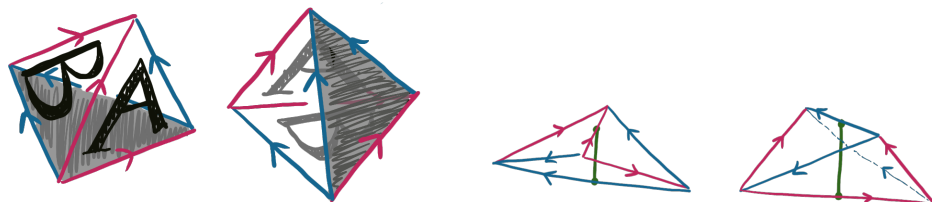


FIGURE 23. (left) The veering triangulation for the figure eight knot complement. T_1 is on the left and T_2 is on the right. (right) and the directed flow cycle through the edges in green.

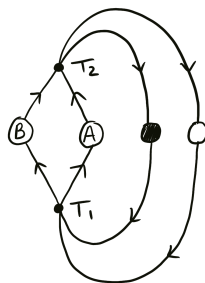


FIGURE 24. The dual graph for the veering triangulation of $S^3 \setminus K_{4_1}$.

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