All knotted up

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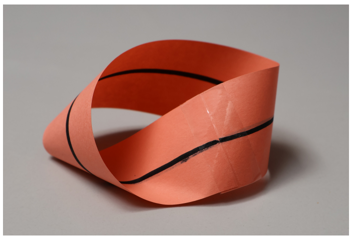
Take a long paper strip, twist one end 180°, and attach the short ends with some tape. This produces a *Möbius loop*.



If have spent some time in the math communication world, you might know the result of cutting this shape along the center line. But, if this is new for you, take a moment and try the experiment for yourself! (Or just watch the video below.)

[insert one-twist.mp4]

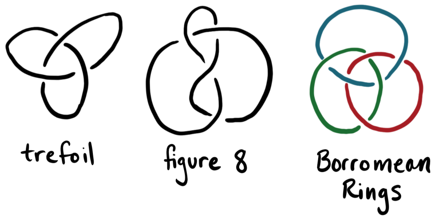
Why do we get only one loop? My favorite explanation is to view the resulting longer loop as a thickened version of the boundary of the Möbius loop. This experiment tells us that the boundary of the Möbius loop is a single unknotted circle.

Consider repeating this process with more twists. For a Möbius loop with k half-twists, what is the result of cutting it in half? If you want to know the answer in full generality, I made a video on this exact topic a few years ago. I will demonstrate one more example here: a thrice-twisted Möbius loop.

[inset three-twists.mp4]

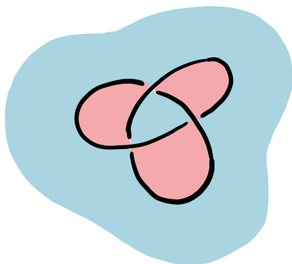
This little activity tells us that the boundary of the thrice-twisted Möbius loop is the *trefoil knot*. Have I surprised you yet?

A mathematical *knot* is an continuous embedding of the circle into $\mathbb{R}^3$. Think of a piece of string with the ends attached. Two knots are the same (*ambiently isotopic*) if you can "wiggle them around" in $\mathbb{R}^3$ to look the same. We can also have *links* which are embeddings of a finite disjoint collection of circles into $\mathbb{R}^3$. I will often abuse terminology and call them all "knots". Some of my favorite examples are below.

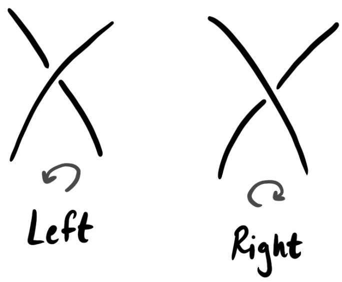


Knots are a bit squirrely in character. For example, mathematicians don't yet know an algorithm to tell if two knots are the same! One way to study knots is to instead study a *Seifert surface* of a knot. This is an (orientable) surface with the given knot (or link) as the boundary. Although the "orientable" condition does have some uses, it doesn't matter for our purposes so we will consider both orientable and non-orientable Seifert surfaces. Our investigation above reveals that the thrice-twisted Möbius loop is a (non-orientable) Seifert surface for the trefoil knot.

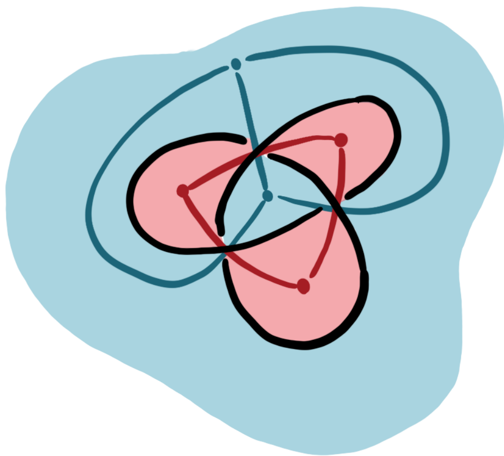
Given a knot, there is a simple algorithm to generate a corresponding Seifert surface (two surfaces, in fact). First draw a planar knot diagram which sections the plane into different faces. We can color these faces red and blue in an alternating way so that no two faces sharing a bounding edge are the same color.



Place a vertex in each red face and connect each pair of vertices once for each crossing between the corresponding faces. Mark these edges L or R for a left-hand or right-hand twist. Repeat with the blue faces. Then we have generated two planar graphs which are *dual* to each other by construction. Each of these graphs encode a Seifert surface. Place a disk at each vertex of the graph and a twisted band with a right or left twist for each edge. We are left with a surface that has the given knot as a boundary.



Examples of left and right hand crossings for the knot diagram. They correspond with twisting a band is the indicated direction.

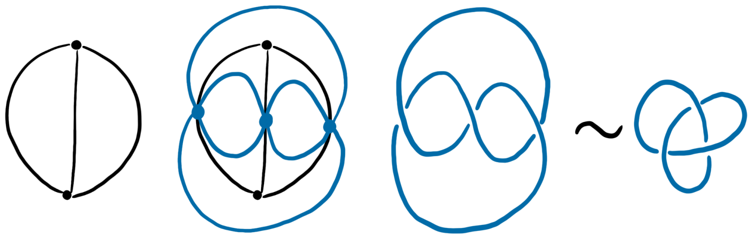


Consider the example above for the trefoil knot. The red graph represents the thrice-twisted Möbius loop; the blue graph provides an alternative Seifert surface. The trefoil is an *alternating knot*, meaning that all its crossings have the same orientation. Thus, we have omitted the left/right labeling for this example.

Last year I became acquainted with Shiying Dong's fantastic work in topological crochet. Shiying constructs Seifert surfaces with natural crochet principles by performing a sort of inverse of the process described above. She starts with a graph— some vertices and edges— and imagines each edge as a "ribbon" with thickness. Then she applies a right-hand twist to each edge and reattaches it. If she starts with the blue graph in the figure above (consisting of two vertices with three edges between them), she obtains the orientable Seifert surface for the trefoil.

A crocheted orientable Seifert surface of the trefoil.

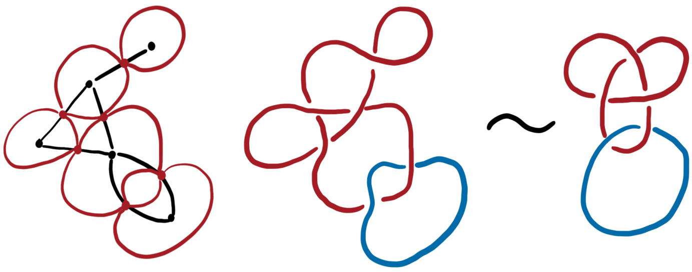
This is the first crochet project that I learned from Shiying and the first that she teaches in her workshops. You can learn to make one for yourself with my tutorial video or the videos on Shiying's channel! But without yarn and a crochet hook on hand, we can perform the same process on a graph using pen and paper.



The graph algorithm performed to obtain the orientable trefoil Seifert surface.

For a general planar graph, the steps are as follows.

1. Draw a planar graph G. It can have double edges and loops, but no edges should cross. The cyclic order of the edges at each vertex is fixed by the embedding into the plane.
2. We will construct a new graph H. Put a vertex of H on each edge of G in a new color.
3. A given edge of G has four "neighboring edges". If the endpoints of edge e are vertices v and w, then the neighboring edges of e are on its right and left sides in the cyclic order at each vertex v and w. Some edges may be double counted.
4. Add an edge in H between two vertices if the corresponding edges in G are neighbors. Each vertex in H should have degree 4.
5. At each vertex of H, draw a right-handed crossing.
6. You are left with a knot or a link!



The graph algorithm performed on a random graph.

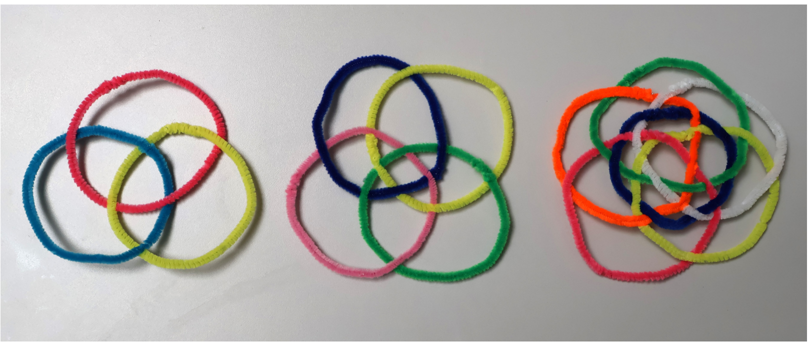
In this process, we have made a universal choice for the handedness of each crossing. This means that all knots and links generated in this way will be alternating. A modification would be to choose the handedness of each crossing individually.

I have really enjoyed trying this on my favorite graphs. Sometimes the resulting knot is surprising and reveals an unexpected connection. In general, it seems difficult to determine which knot or link we obtain—or even the number of link components.

What are the next graphs that you would try? Here, I'll choose the edge graphs of the Platonic solids. Let's start with the tetrahedron. Draw four vertices and connect every pair of vertices with an edge. What link do we get when we apply with the process above?

A crocheted version of the Seifert surface resulting from the tetrahedron.

Try for yourself and you will see that we meet another old friend: the Borromean rings. If we continue with the other platonic solids, we obtain two more woven links, one for the cube/octahedron, and one for the icosahedron/dodecahedron. (Recall that dual graphs will give the same link.) These links can be nicely woven by pipe cleaners; thanks to my friend Elliot Kienzle for pointing that out!



Some other great examples to try are the wheel graphs and the edge graphs of prisms. What corresponding graphs and knots can you find? Whether you engage with this question by crocheting surfaces or by doodling graphs, there are many surprises and delights to be found.

The Seifert surface generated using the edge graph of the cube (left) and the surface from the edge graph of the tetrahedron (right).